08.11.23

2.4.1 Déterminer les applications injectives, surjectives ou bijectives.

g) $f_7 : \mathbb{N} \to \mathbb{N}$, $x \mapsto x^2$ injective $\qquad f_7(4) = 16$

h) $f_8 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ $\qquad f_8(4) = 16$

g) • Soit $a, b \in \mathbb{N}$

$$f_7(a) = f_7(b) \iff a^2 = b^2 \iff a^2 - b^2 = 0 \iff \underbrace{(a-b)}_{a=b}\overbrace{(a+b)}^{a=-b \text{ impossible}} = 0$$

$\Rightarrow f_7$ injective

• $f(a) = 3 \Rightarrow a^2 = 3 \Rightarrow a = \sqrt{3} \notin \mathbb{N}$ $\qquad f_7$ non surjective

h) • $f_7(1) = f_7(-1) = 1$ $\qquad$ non injective

• $y \in \mathbb{R}$ $\qquad 4 \in \mathbb{R}$ $\qquad 5 \in \mathbb{R}$

$f(\sqrt{y}) = y$ $\qquad f(2) = 4$ $\qquad f(\sqrt{5}) = 5$

Si $y < 0$, c'est impossible $\qquad$ non surjective

j) $\begin{array}{rccc} f_{10} : & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^3 \end{array}$

l) $\begin{array}{rccc} f_{12} : & \mathbb{R} & \to & \mathbb{R}_+ \\ & x & \mapsto & x^2 \end{array}$

k) $\begin{array}{rccc} f_{11} : & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^3 - x^2 \end{array}$

m) $\begin{array}{rccc} f_{13} : & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & x & \mapsto & x^2 \end{array}$

	Injectif	Surjectif
f_{10}		
f_{11}		
f_{12}		
f_{13}	✓	✓

$a, b \in \mathbb{R}_+$

m) • $f_{13}(a) = f(b) \Leftrightarrow a^2 = b^2 \Leftrightarrow a = b$

$\Rightarrow$ injectif

• $y \in \mathbb{R}_+$

$f_{13}(\sqrt{y}) = y$

$f_{13}(\sqrt{13}) = 13$

$\Rightarrow$ surjectif

j) $f_{10} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$

l) $f_{12} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}_+ \\ x & \mapsto & x^2 \end{array}$

k) $f_{11} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 - x^2 \end{array}$

m) $f_{13} : \begin{array}{ccc} \mathbb{R}_+ & \to & \mathbb{R}_+ \\ x & \mapsto & x^2 \end{array}$

	Injectif	Surjectif
f_{10}	✓	✓
f_{11}	✗	✓
f_{12}	✗	✓
f_{13}	✓	✓

$a, b \in \mathbb{R}_+$

m) • $f_{13}(a) = f(b) \Leftrightarrow a^2 = b^2 \Leftrightarrow a = b$

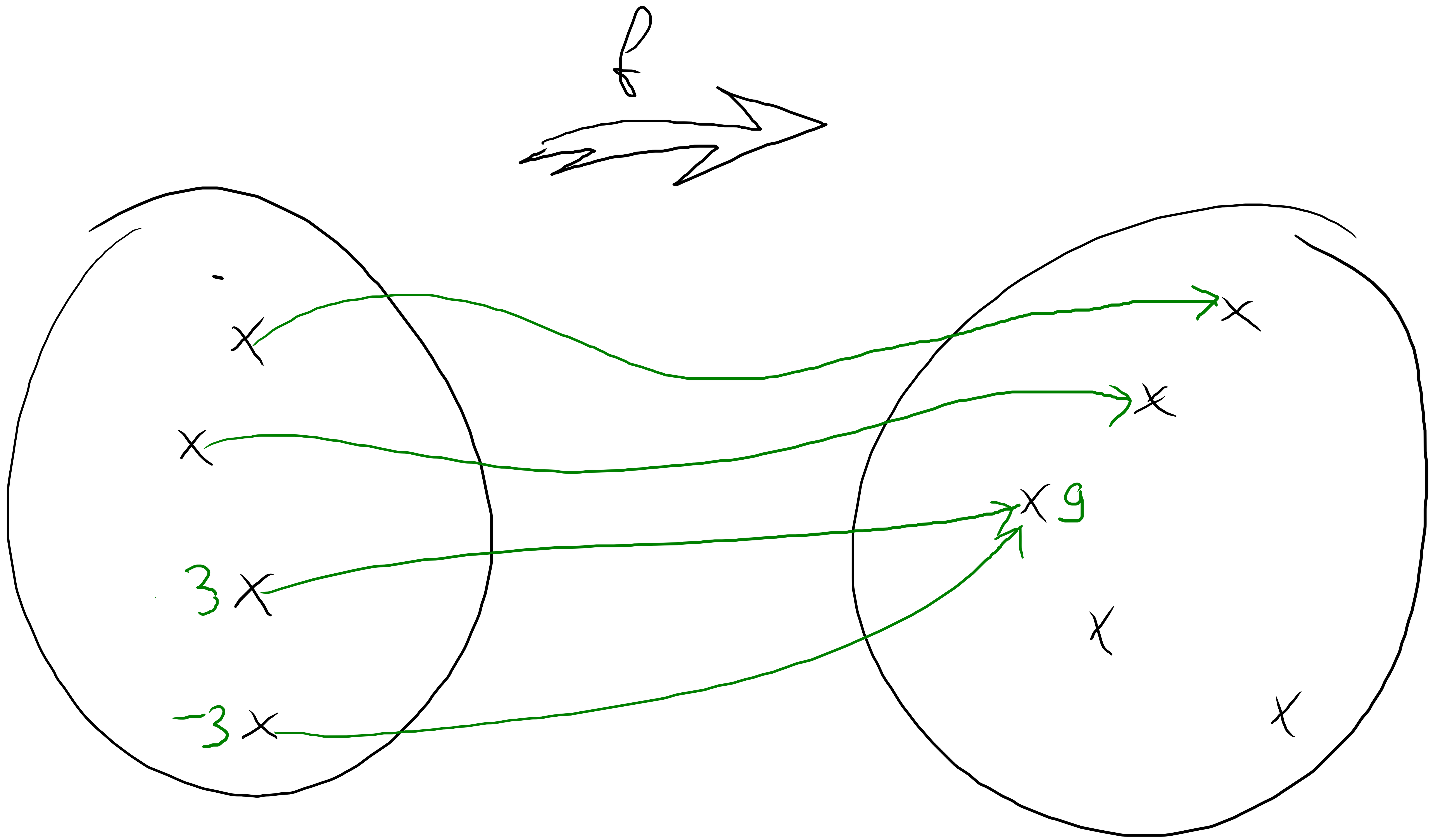
$\Rightarrow$ injectif

• $y \in \mathbb{R}_+$

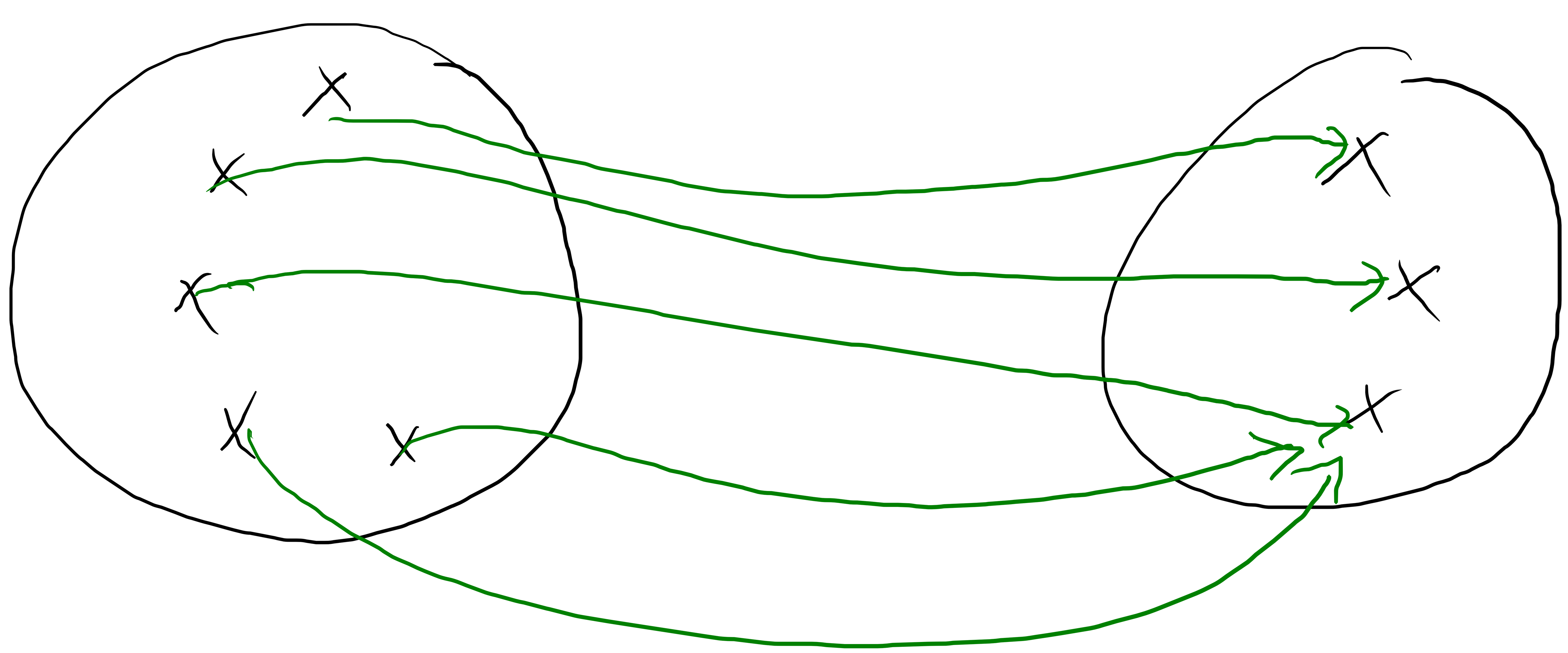
$f_{13}(\sqrt{y}) = y$

$f_{13}(\sqrt{13}) = 13$

$\Rightarrow$ surjectif



$$f: \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto n^2$$



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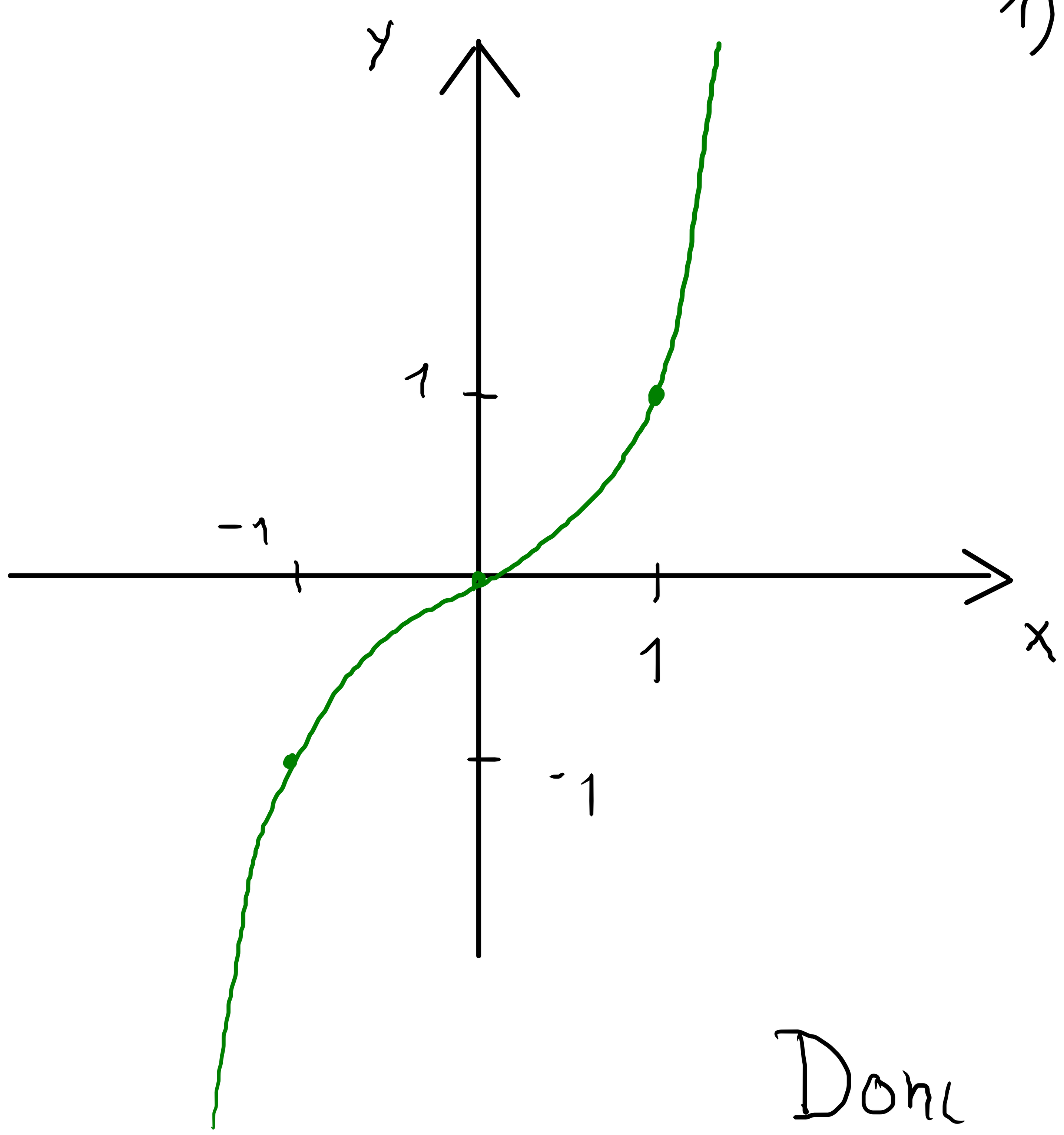
j) $f_{10} : \mathbb{R} \to \mathbb{R}$
$x \mapsto x^3$



1) Injection

$a, b \in \mathbb{R}$, $f_{10}(a) = f_{10}(b) \Rightarrow a^3 = b^3 \Rightarrow a = b$

donc f_{10} est injective

2) Soit $y \in \mathbb{R}$, $f_{10}(\sqrt[3]{y}) = y$, avec $\sqrt[3]{y} \in \mathbb{R}$

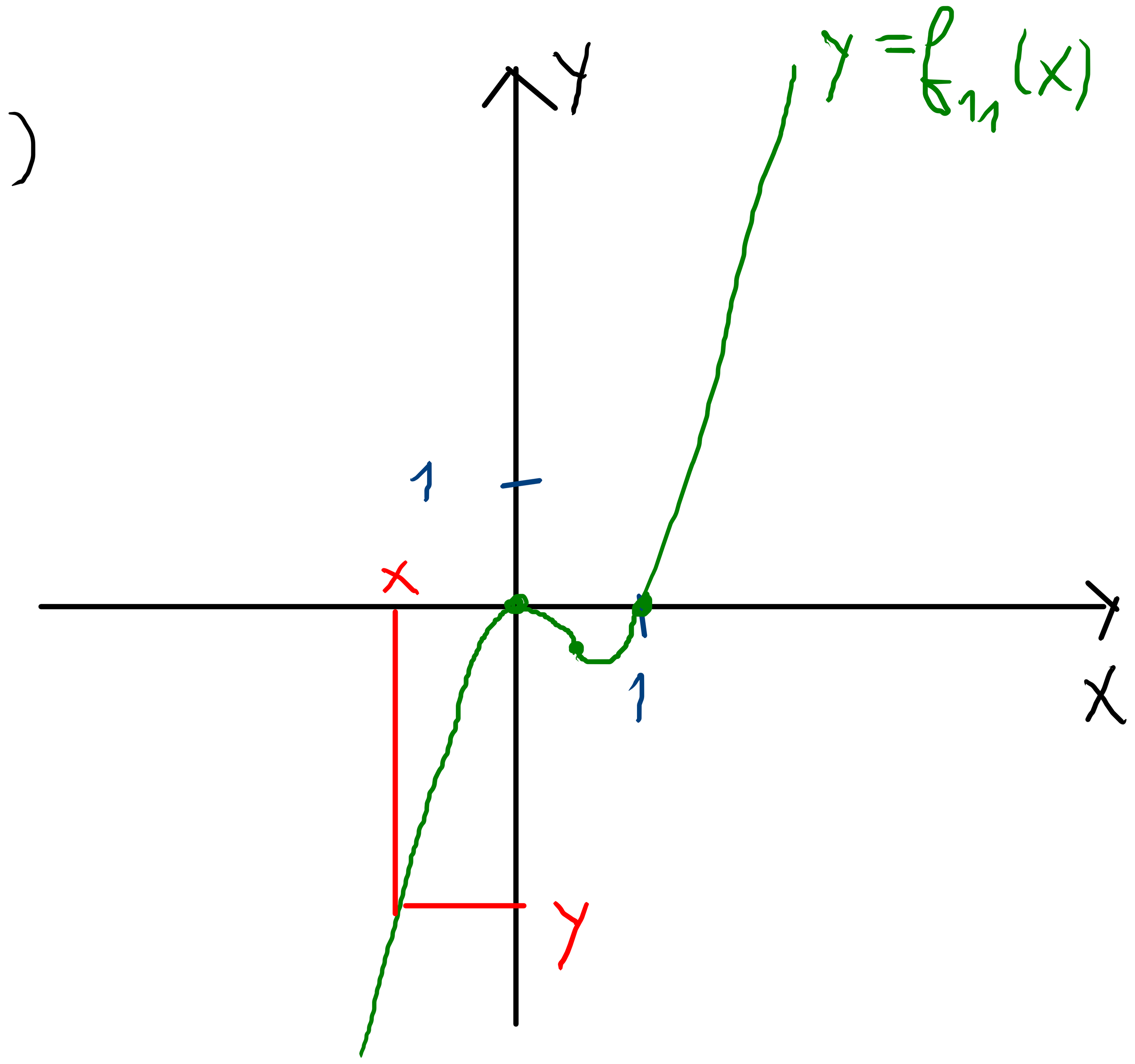
Donc f_{10} est surjective et ainsi bijective.

k) $$f_{11} : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 - x^2 \end{array}$$

1) Injective :

$$\left.\begin{array}{l} f_{11}(0) = 0 \\ f_{11}(1) = 0 \end{array}\right\} \text{ pas injective}$$

2)



$f_{11}(0,5) = 0,125 - 0,25 = -0,125$

La fonction est surjective, le graphique le montre !

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ tel $f(x) = y$

1) $$f_{12} : \begin{array}{rcl} \mathbb{R} & \to & \cancel{\mathbb{R}}_+ \\ x & \mapsto & x^2 \end{array}$$

- Injective : $f(-3) = f(3) = 9$

  pas injectif.

- Surjectif : soit $y \in \cancel{\mathbb{R}}_+$, alors $f(\sqrt{y}) = y$. Donc surjectif

Application réciproque

Soit une fonction réelle bijective.

$$\begin{aligned} f : A &\longrightarrow B \\ x &\longmapsto f(x) \end{aligned}$$

Il existe une application f^r telle que

$$f_r \circ f\,(x) = x \quad , \quad \forall\, x \in A$$

$$f \circ f_r\,(y) = y \quad , \quad \forall\, y \in B$$

$$\begin{aligned} f^r : B &\longrightarrow A \\ y &\longmapsto {}^r f(y) \end{aligned}$$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f^r} & A \\ x & \longmapsto & y & \longmapsto & x \end{array}$$

$$\begin{array}{ccccc} B & \xrightarrow{f^r} & A & \xrightarrow{f} & B \\ y & \longmapsto & x & \longmapsto & y \end{array}$$

Soit l'application bijective

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto 3x + 2$$

Déterminons l'application réciproque.

Par exemple $f(5) = 17$, $f^r(17) = 5$.

Soit $a \in \mathbb{R}$, $f(x) = a$.

$$3x + 2 = a \quad \Big| \; -2$$
$$3x = a - 2 \quad \Big| \; \div 3$$
$$x = \frac{a-2}{3}$$

Donc $f^r : \mathbb{R} \longrightarrow \mathbb{R}$

$$y \longmapsto \frac{y}{3} - \frac{2}{3}$$

2.4.2 Définir l'application réciproque des bijections.

a) $f_1 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}$

b) $f_2 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & 3\,x \end{array}$

c) $f_3 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & 2\,x+3 \end{array}$

d) $f_4 : \begin{array}{ccc} \mathbb{R}^* & \to & \mathbb{R}^* \\ x & \mapsto & \dfrac{1}{x} \end{array}$

e) $f_5 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$

f) $f_6 : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ x & \mapsto & x-3 \end{array}$

g) $f_7 : \begin{array}{ccc} \mathbb{R}-\{1\} & \to & \mathbb{R}-\{2\} \\ x & \mapsto & \dfrac{2\,x+1}{x-1} \end{array}$

h) $f_8 : \begin{array}{ccc} \mathbb{R}-\{1\} & \to & \mathbb{R}-\{1\} \\ x & \mapsto & \dfrac{x+1}{x-1} \end{array}$

a) $f_1^r(y) = y$

b) $f_2^r(y) = \frac{1}{3}y$ $\qquad 3x = y \Rightarrow x = \frac{1}{3}y$

c) $2x + 3 = y$ $\qquad f^r(y) = \frac{1}{2}y - \frac{3}{3}$

$2x = y - 3$

$x = \frac{y}{2} - \frac{3}{2}$

d) $\frac{1}{x} = y$ $\qquad f^r(y) = \frac{1}{y}$

$1 = xy$

$\frac{1}{y} = x$

e) $x^3 = y$ $\qquad f^r(y) = \sqrt[3]{y}$

$x = \sqrt[3]{y}$

f) $f^r(y) = x+3$

g) $\begin{aligned} f_7 \ : \ \mathbb{R}-\{1\} &\to \mathbb{R}-\{2\} \\ x &\mapsto \frac{2\,x+1}{x-1} \end{aligned}$

h) $\begin{aligned} f_8 \ : \ \mathbb{R}-\{1\} &\to \mathbb{R}-\{1\} \\ x &\mapsto \frac{x+1}{x-1} \end{aligned}$

g) $y \in \mathbb{R}-\{2\}$, $\exists\, x \in \mathbb{R}-\{1\}$ tq

$$\frac{2x+1}{x-1} = y$$

$$2x+1 = y(x-1)$$

$$2x+1 = yx-y$$

$$2x-yx = -y-1$$

$$x(2-y) = -y-1$$

$$\frac{2x+1}{x-1} = 4 \quad \Big| \cdot (x-1)$$

$$2x+1 = 4(x-1) \quad \Big| \text{ CL}$$

$$2x+1 = 4x-4 \quad \Big| -4x-1$$

$$-2x = -5 \quad \Big| \div(-2)$$

$$x = \frac{5}{2}$$

$$x = \frac{-y-1}{2-y} = \frac{-(y+1)}{-(y-2)} = \frac{y+1}{y-2}$$

$$\begin{aligned} f^r : \mathbb{R}-\{2\} &\longrightarrow \mathbb{R}-\{1\} \\ y &\longmapsto \frac{y+1}{y-2} \end{aligned}$$

$f_7(x) = \frac{2x + 1}{x - 1}$

$g_7(x) = \frac{x + 1}{x - 2}$

f: y = x

A = (1.99, 5.02)

A' = (5.02, 1.99)

g: x = 1.99

h: y = 5.02

i: x = 5.02

j: y = 1.99

B = (1.99, 0)

C = (0, 1.99)

D = (5.02, 0)

E = (0, 5.02)