31.01.24

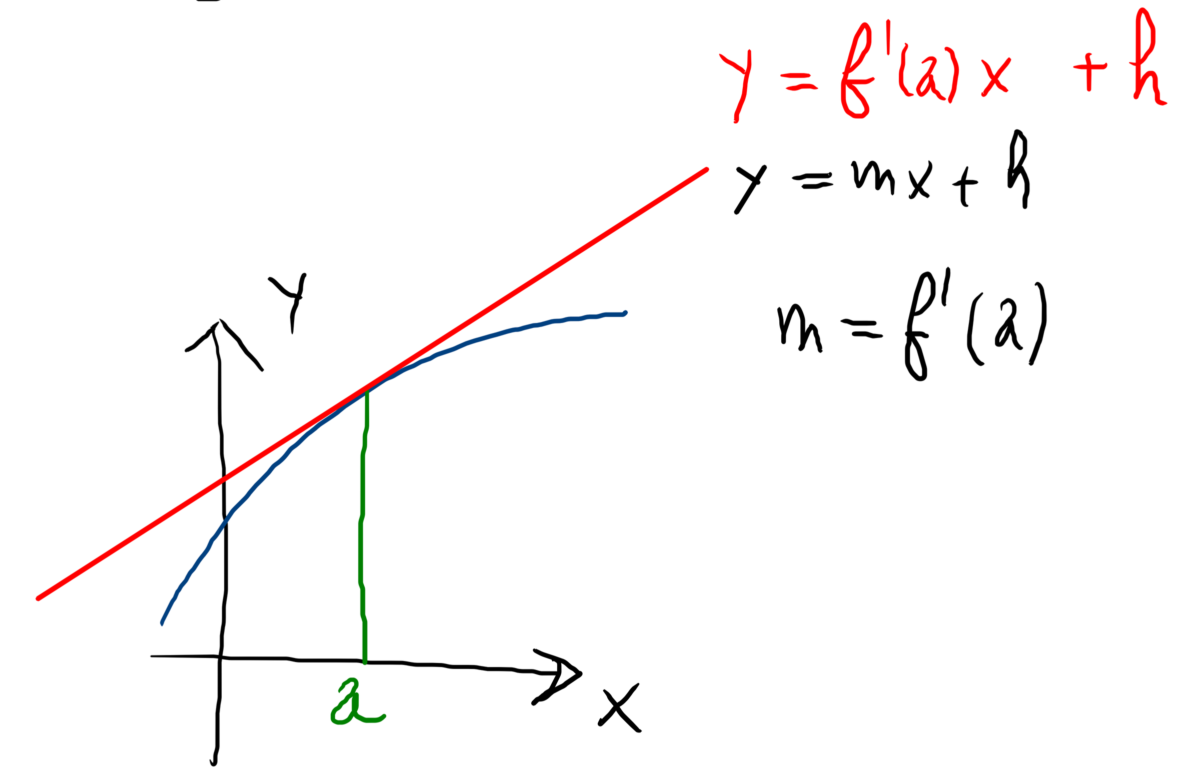
2.9.1 Calculer $f'(x)$, à partir de la définition de la dérivée, si :

e) $f(x) = \dfrac{4x-1}{x+1}$

f) $f(x) = \sqrt{x-3}$

e) $ED(f) = \mathbb{R} - \{-1\}$, $EC(f) = \mathbb{R} - \{-1\}$

$$\boxed{\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}}$$



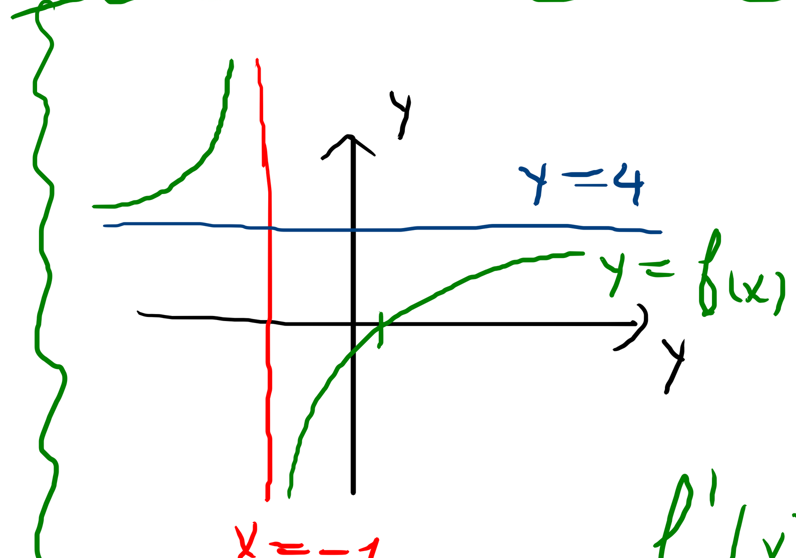
$$f'(a) = \lim_{x \to a} \frac{\frac{4x-1}{x+1} - \frac{4a-1}{a+1}}{x-a} = \lim_{x \to a} \frac{\frac{(4x-1)(a+1) - (4a-1)(x+1)}{(x+1)(a+1)}}{x-a}$$

$$= \lim_{x \to a} \frac{(\cancel{4ax} + 4x - a - \cancel{1}) - (\cancel{4ax} + 4a - x - \cancel{1})}{(x+1)(a+1)(x-a)} = \lim_{x \to a} \frac{4x - a - 4a + x}{(x+1)(a+1)(x-a)}$$

$$= \lim_{x \to a} \frac{5x - 5a}{(x+1)(a+1)(x-a)} = \lim_{x \to a} \frac{5\cancel{(x-a)}}{(x+1)(a+1)\cancel{(x-a)}} = \lim_{x \to a} \frac{5}{(x+1)(a+1)}$$

$$= \frac{5}{(a+1)^2} \qquad EC(f') = \mathbb{R} - \{-1\}$$

$$\Rightarrow f'(x) = \frac{5}{(x+1)^2}$$



$f'(x) > 0 \quad \forall x \in ED(f)$

f est toujours croissante

f) $f(x) = \sqrt{x-3}$

$ED(f) = [3 ; +\infty[$ $\qquad$ $EC(f) = [3 ; +\infty[$

défini dans un

$$f'(a) = \lim_{x \to a} \frac{\sqrt{x-3} - \sqrt{a-3}}{x-a} = \lim_{x \to a} \frac{\sqrt{x-3} - \sqrt{a-3}}{x-a} \cdot \boxed{\frac{\sqrt{x-3} + \sqrt{a-3}}{\sqrt{x-3} + \sqrt{a-3}}}$$

voisinage de $x = a$

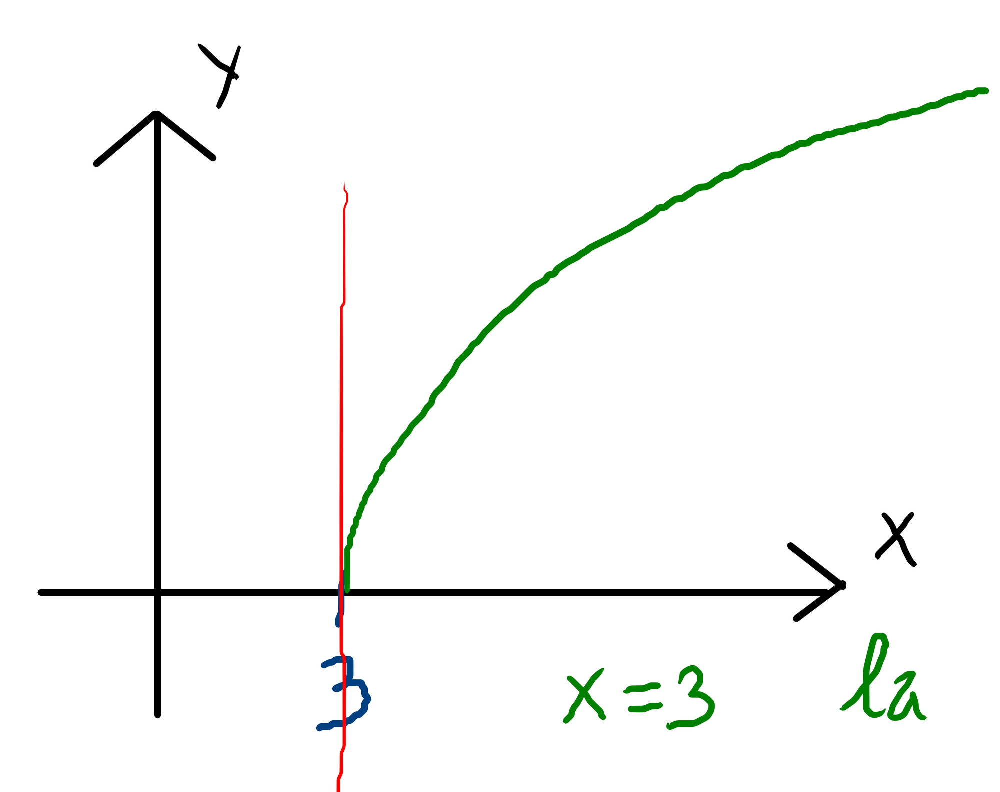
$$= \lim_{x \to a} \frac{(x-3)-(a-3)}{(x-a)(\sqrt{x-3}+\sqrt{a-3})} = \lim_{x \to a} \frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x-3}+\sqrt{a-3})}$$

$$= \lim_{x \to a} \frac{1}{\sqrt{x-3} + \sqrt{a-3}} = \frac{1}{\sqrt{a-3}+\sqrt{a-3}} = \frac{1}{2\sqrt{a-3}}$$

$$f'(x) = \frac{1}{2\sqrt{x-3}} \qquad ED(f') = EC(f') =]3 ; +\infty[$$

Que se passe-t-il en $x = 3$?

$$\lim_{\substack{x \to 3 \\ >}} f'(x) = +\infty$$



$x=3$ la tangente est verticale

Dérivée de fonctions usuelles

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	
a	0	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	
x	1	$\frac{1}{x}$	$-\frac{1}{x^2}$	
x^n	nx^{n-1}	$\lvert x\rvert$	$\operatorname{sgn}(x)$	$x \neq 0$
e^x	e^x	$\ln(x)$	$\frac{1}{x}$	
a^x	$a^x \ln(a)$	$\log_a(x)$	$\frac{1}{x\ln(a)}$	
$\sin(x)$	$\cos(x)$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	
$\cos(x)$	$-\sin(x)$	$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\arctan(x)$	$\frac{1}{1+x^2}$	
$\cot(x)$	$-\frac{1}{\sin^2(x)} = -1 - \cot^2(x)$	$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$	
$\sinh(x)$	$\cosh(x)$	$\operatorname{arsinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$	
$\cosh(x)$	$\sinh(x)$	$\operatorname{arcosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$	
$\tanh(x)$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$	$\operatorname{artanh}(x)$	$\frac{1}{1-x^2}$	$\lvert x\rvert < 1$
$\coth(x)$	$-\frac{1}{\sinh^2(x)} = 1 - \coth^2(x)$	$\operatorname{arcoth}(x)$	$\frac{1}{1-x^2}$	$\lvert x\rvert > 1$

Formules de dérivation

1) $f(x) = x^n$, pour $n \in \mathbb{N}^*$

- $(x)' = 1$
- $(x^2)' = 2x$
- $f(x) = x^3$: $f'(a) = \lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a} = \lim\limits_{x \to a} \dfrac{\cancel{(x-a)}(x^2 + ax + a^2)}{\cancel{x-a}} = 3a^2$

  $(x^3) = 3x^2$

- $f(x) = x^n$: $f'(a) = \lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = \lim\limits_{x \to a} \left(x^{n-1} + a x^{n-2} + \ldots + a^{n-1} \right) = n a^{n-1}$

$$\begin{array}{c|cccccc|c} & 1 & \cdot & \cdot & & \cdot\cdot & & -a^n \\ a & & a & a^2 & & \cdots & a^{n-1} & a^n \\ \hline & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & 0 \end{array}$$

$(x^n) = n x^{n-1}$

2) Par extension et sans démonstration :

$n \in \mathbb{Q}^*$ $\quad (x^n)' = n \cdot x^{n-1}$

$n \in \mathbb{R}^*$ $\quad (x^n)' = n \cdot x^{n-1}$

Exemple :

1) $(x^{18})' = 18 x^{17}$

2) $\left(x^{\frac{2}{3}}\right)' = \frac{2}{3} x^{-\frac{1}{3}}$; $\left(\sqrt[3]{x^2}\right)' = \dfrac{2}{3\sqrt[3]{x}}$

3) $(x^{-4})' = -4x^{-5}$; $\left(\dfrac{1}{x^4}\right)' = \dfrac{-4}{x^5}$

3) $(\sin(x))' = ?$

$f(x) = \sin(x)$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\sin(a+h) - \sin(a)}{h}$$

CRM : $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$$= \lim_{h\to 0} \frac{\sin(a)\cos(h) + \sin(h)\cos(a) - \sin(a)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(a)\left[\cos(h) - 1\right]}{h} + \lim_{h\to 0} \frac{\cos(a)\,\sin(h)}{h}$$

2.6.9

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

$$= \sin(a) \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h) - 1}{h}}_{} + \cos(a)\cdot \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{}$$

$$= 0 + \cos(a) \cdot 1$$

$$\Rightarrow \boxed{(\sin(x))' = \cos(x)}$$

4) $\boxed{(\cos(x))' = -\sin(x)}$

5) $\boxed{(f(x) + g(x))' = f'(x) + g'(x)}$

$$(x^3 + \sin(x))' = 3x^2 + \cos(x)$$

$$\left[f(a) + g(a)\right]' = \lim_{x\to a} \frac{\left[f(x) + g(x)\right] - \left[f(a) + g(a)\right]}{x - a}$$

$$= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} + \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = f'(a) + g'(a)$$

6) $$\left(f(x) \cdot g(x) \right)' = f'(x)\, g(x) + f(x)\, g'(x)$$

$$\left(f(a) \cdot g(a) \right)' = \left((fg)(a) \right)' = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) - f(x)g(a) + f(x)\,g(a) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)\left[g(x) - g(a) \right]}{x - a} + \lim_{x \to a} \frac{g(a)\left[f(x) - f(a) \right]}{x - a}$$

$$= f(a) \cdot g'(a) + g(a) \cdot f'(a)$$

$$\left(\underbrace{x^3}_{f} \cdot \underbrace{\sin(x)}_{g} \right)' = 3x^2 \cdot \sin(x) + x^3 \cos(x) = \underline{x^2 \left(3\sin(x) + x\cos(x) \right)}$$

7) $$\boxed{\left(\frac{1}{f(x)}\right)' = \frac{-f(x)}{[f(x)]^2}}$$

$$\left(\frac{1}{f(a)}\right)' = \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} = \lim_{h\to 0} \frac{\frac{f(a)-f(a+h)}{f(a+h)\cdot f(a)}}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{-(f(a+h)-f(a))}{h}}_{-f'(a)} \cdot \underbrace{\frac{1}{f(a+h)\,f(a)}}_{\frac{1}{(f(a))^2}}$$

$$= -f'(a)\cdot\frac{1}{(f(a))^2} = \frac{-f'(a)}{(f(a))^2}$$

$$\left(\frac{1}{x^7}\right)' = \frac{-7x^6}{x^{14}} = \frac{-7}{x^8} \quad ; \quad (x^{-7})' = -7x^{-8}$$

$$\frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \underset{\substack{\uparrow \\ \text{photomath}}}{=} -\cot(x)\csc(x)$$

$$= \frac{-\cos(x)}{\sin^2(x)} = -\underbrace{\frac{\cos(x)}{\sin(x)}}_{\cot(x)} \cdot \underbrace{\frac{1}{\sin(x)}}_{\csc(x)}$$

8) $$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' \underset{6}{=} f'(x)\,\frac{1}{g(x)} + f(x)\cdot\frac{-g'(x)}{g^2(x)}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$