

Exercice 1.2.6

$$\sum_{k=1}^n (2k-1)^2 = 1^2 + 3^2 + 5^2 + 7^2 + \dots + n^2$$

n impair

k	1	2	3	4	5	6
Σ	1	10	35	84	165	286

Supposons que $\sum_{k=1}^n (2k-1)^2 = \frac{an^3 + bn^2 + cn + d}{1}$

$$\begin{array}{l} n=1 \\ n=2 \\ n=3 \\ n=4 \end{array} \left\{ \begin{array}{l} a+b+c+d = 1 \\ 8a+4b+2c+d = 10 \\ 27a+9b+3c+d = 35 \\ 64a+16b+4c+d = 84 \end{array} \right. \begin{array}{l} \cdot (-1) \\ \cdot 1 \\ \cdot (-1) \\ \cdot 1 \end{array} \left| \begin{array}{l} \cdot (-1) \\ \cdot (-1) \\ \cdot 1 \\ \cdot (-1) \end{array} \right| \begin{array}{l} \cdot (-1) \\ \cdot 1 \\ \cdot (-1) \\ \cdot 1 \end{array}$$

$$\begin{array}{l} \left\{ \begin{array}{l} 7a+3b+c = 9 \\ 19a+5b+c = 25 \\ 37a+7b+c = 49 \end{array} \right. \begin{array}{l} \cdot (-1) \\ 1 \\ \cdot (-1) \end{array} \left| \begin{array}{l} \cdot 1 \\ \cdot 1 \\ \cdot (-1) \end{array} \right. \end{array} \quad \begin{array}{l} \left\{ \begin{array}{l} 12a+2b = 16 \\ 18a+2b = 24 \end{array} \right. \end{array} \quad \begin{array}{l} \left\{ \begin{array}{l} 6a+b=8 \\ 9a+b=12 \end{array} \right. \\ 3a=4 \\ a=\frac{4}{3} \end{array}$$

$$a = \frac{4}{3}, \quad b = 0, \quad c = -\frac{1}{3}, \quad d = 0$$

$$\frac{4}{3}n^3 - \frac{1}{3}n = \frac{4n^3 - n}{3} \quad \text{semble \u00eatre la formule cherch\u00e9e.}$$

D\u00e9montrons par r\u00e9currence que

$$\sum_{k=1}^n (2k-1)^2 = \frac{4n^3 - n}{3} = \frac{n(4n^2 - 1)}{3} = \frac{n(2n-1)(2n+1)}{3}$$

① vraie pour $n=1$: $1 \stackrel{?}{=} \frac{4 \cdot 1 - 1}{3} = 1$ OK

② Supposons le résultat vrai pour n et démontrons-le pour $n+1$:

$$\begin{aligned} \sum_{k=1}^{n+1} (2k-1)^2 &= \frac{4n^3 - n}{3} + (2(n+1)-1)^2 \\ &= \frac{4n^3 - n}{3} + (2n+1)^2 \\ &= \frac{(2n+1)(2n-1) \cdot n}{3} + (2n+1)^2 \\ &= (2n+1) \left[\frac{n(2n-1)}{3} + 2n+1 \right] = (2n+1) \frac{2n^2 - n + 6n + 3}{3} \\ &= (2n+1) \frac{2n^2 + 5n + 3}{3} = \frac{(2n+1)(2n+3)(n+1)}{3} \\ &= \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3} \end{aligned}$$

Comme la relation est vérifiée pour $n+1$, elle est vérifiée pour tout n .