1.3.15

a) $\displaystyle\int_1^2 \frac{x}{x+6}\,dx$

Pour intégrer une fraction rationnelle, il faut que le degré du polynôme au numérateur soit strictement inférieur au degré du polynôme du dénominateur.
Si ce n'est pas le cas, on effectue la division euclidienne.

$$\begin{array}{r|l} x & x+6 \\ \cline{2-2} -\;\underline{x+6} & 1 \\ -6 & \end{array} \qquad \frac{x}{x+6} = 1 + \frac{-6}{x+6}$$

$$\int \frac{x}{x+6}\,dx = \int \left(1 + \frac{-6}{x+6}\right)dx = \underbrace{\int 1\cdot dx}_{x} \underbrace{-6\int \frac{1}{x+6}\,dx}_{-\,6\ln|x+6|}$$

$$\Rightarrow \int_1^2 \frac{x}{x+6} = \Big[x - 6\ln|x+6|\Big]_1^2 = \big(2-6\ln(8)\big) - \big(1-6\ln(7)\big)$$

$$= 1 + 6\big(\ln(7) - \ln(8)\big) = \underline{1+6\ln\left(\tfrac{7}{8}\right)}$$

b) $\displaystyle\int_0^4 \sqrt{x}\,(x+2)\,dx$

$$\int x^{\frac{1}{2}}(x+2)\,dx = \int \left(x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right)dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$$

$$\int_0^4 \sqrt{x}\,(x+2)\,dx = \frac{2}{5}\sqrt{x^5} + \frac{4}{3}\sqrt{x^3}\,\Bigg|_0^4 = \frac{2}{5}\left(\sqrt{4}\right)^5 + \frac{4}{3}\left(\sqrt{4}\right)^3 - 0$$

$$= \frac{2}{5}\cdot 32 + \frac{4}{3}\cdot 8 = \frac{352}{15}$$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \underbrace{\sin^5(x)}_{u^5} \cdot \underbrace{\cos(x)}_{u'}\,dx = \frac{1}{6}\sin^6(x)\Big|_0^{\frac{\pi}{2}} = \frac{1}{6}$

d) $\displaystyle\int_2^{\sqrt{20}} 3x\sqrt{(x^2+5)}\,dx = \left(\sqrt{x^2+5}\right)^3\Big|_2^{\sqrt{20}} = \left(\sqrt{25}\right)^3 - \left(\sqrt{9}\right)^3 = 5^3 - 3^3 = 98$

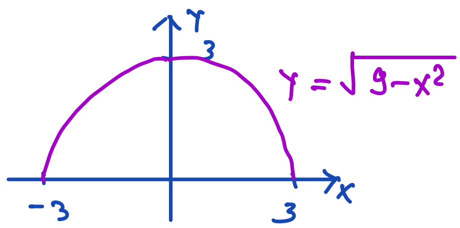
$$\int 3x\,(x^2+5)^{\frac{1}{2}}\,dx = (x^2+5)^{\frac{3}{2}} + c$$

candidat : $K\,(x^2+5)^{\frac{3}{2}}$

(candidat)' : $K\cdot\frac{3}{2}(x^2+5)^{\frac{1}{2}}\cdot 2x = 3K\,(x^2+5)^{\frac{1}{2}} \Rightarrow K = 1$

e) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

$f(x) = \sqrt{9-x^2}$, $-3 \leq x \leq 3$

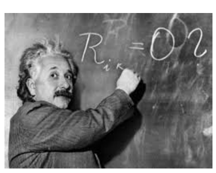


Si $y = \sqrt{9-x^2}$, alors $0 \leq y \leq 3$.

On a $y^2 = 9 - x^2$ et $x^2 + y^2 = 9$, c'est un demi-cercle de centre $(0,0)$ et de rayon 3.

Donc $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{2}\pi\, 3^2 = 4{,}5\,\pi$

f) $\displaystyle\int_2^5 \frac{5x-2}{x^2-x}\,dx$



Ici, il faut décomposer la fraction comme la somme de deux fractions " plus simples " !

$$\frac{5x-2}{x^2-x} = \frac{5x-2}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1} = \frac{a(x-1)+bx}{x(x-1)}$$

$$= \frac{ax+bx-a}{x(x-1)} = \frac{(a+b)x-a}{x(x-1)}$$

Donc $\begin{cases} a+b = 5 \\ -a = -2 \end{cases} \Leftrightarrow \begin{cases} a = 2 \\ b = 3 \end{cases}$

$$\int_2^3 \frac{5x-2}{x^2-x}\,dx = \int_2^3 \left(\frac{2}{x} + \frac{3}{x-1}\right)dx = 2\ln(x)\Big|_2^3 + 3\ln(x-1)\Big|_2^3$$

$$= 2\ln(3) - 2\ln(2) + 3\ln(2) - \underbrace{3\ln(1)}_{0}$$

$$= \ln(9) + \ln(8) - \ln(4)$$

$$= \ln\left(\frac{9\cdot 8}{4}\right) = \ln(18)$$