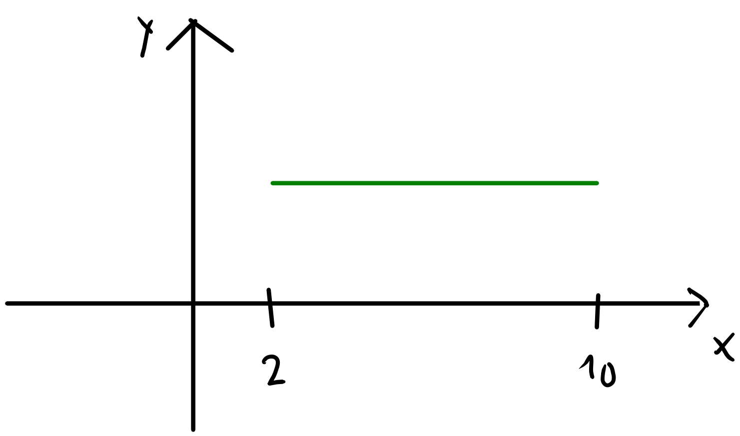
Solide de révolution

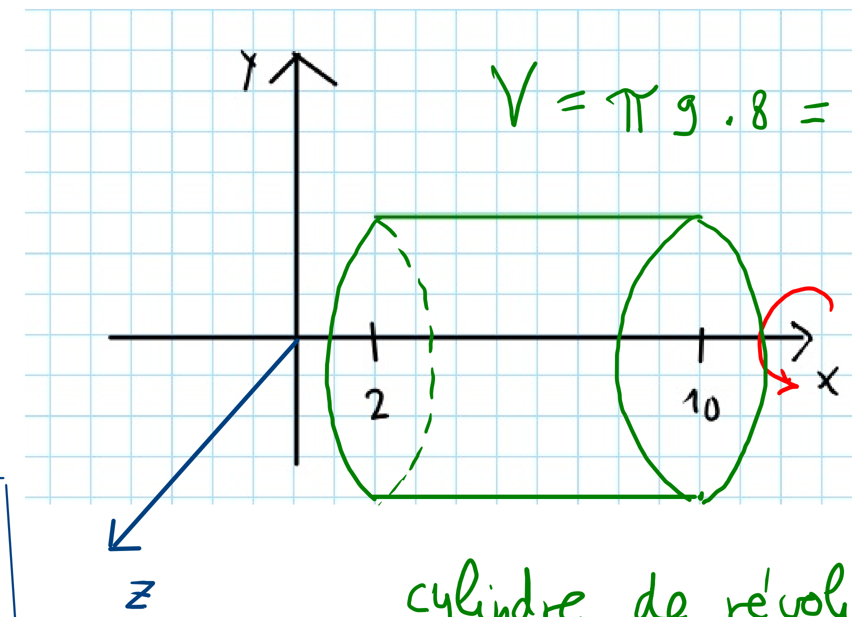
Soit $y = f(x)$ une fonction intégrable sur un intervale $[a, b]$.

En faisant tourner la courbe $y = f(x)$ autour de Ox, on obtient un solide de révolution



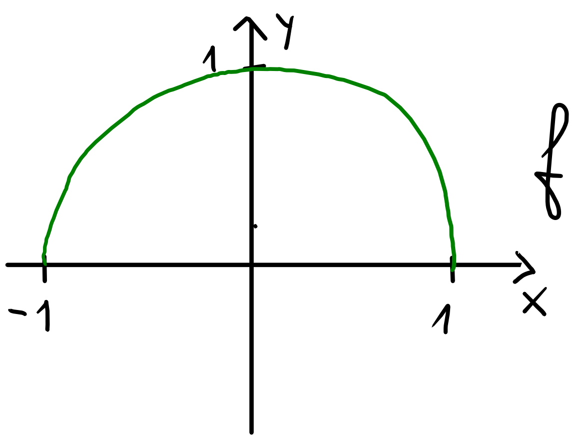
$f(x) = 3$

$[2; 10]$



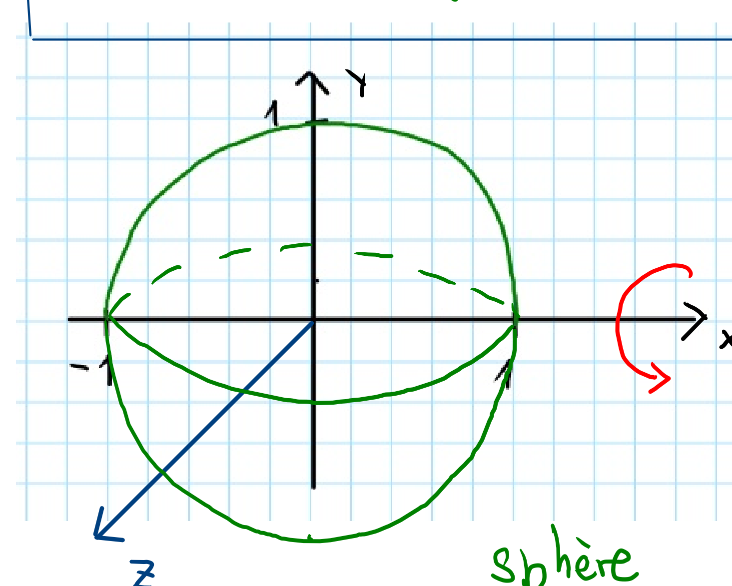
$V = \pi \, 9 \cdot 8 = 72\pi$

cylindre de révolution



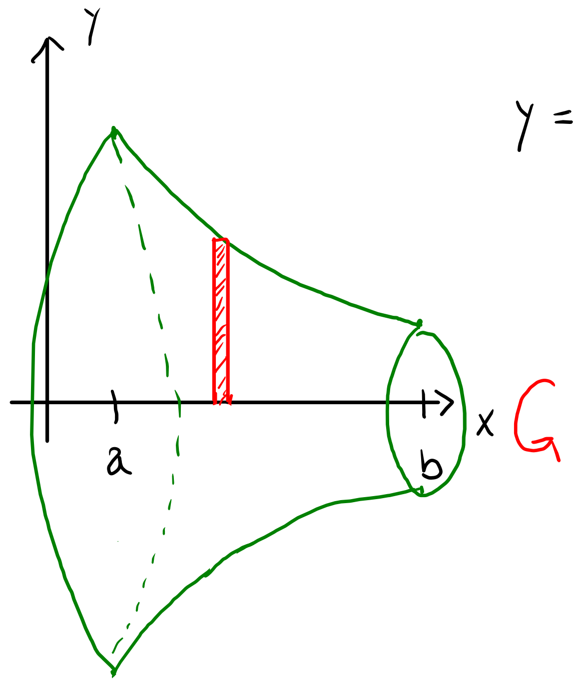
$f(x) = \sqrt{1 - x^2}$

$[-1; 1]$



$V = \frac{4}{3}\pi$

sphère



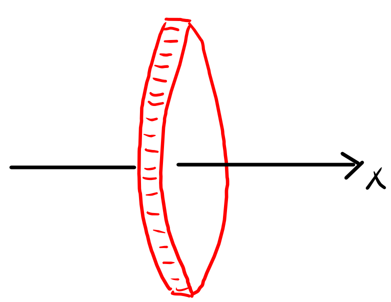
$y = f(x)$ sur $[a,b]$

On découpe $[a,b]$ en n sous-intervalles de même largeur $[a = x_0 ; x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n = b]$

La largeur est $\Delta x = \dfrac{b-a}{n}$

Pour chaque $i = 0, 1, \ldots, n-1$, on dessine un rectangle comme base le segment $x_i \, x_{i+1}$ et comme hauteur $f(x_i)$.

Lorsqu'ils tourneront autour de Ox, chacun de ces rectangles va définir un cylindre de volume



$$V_i = \underbrace{\pi \left(f(x_i) \right)^2}_{\text{base}} \cdot \Delta x$$

$$V = \lim_{n \to +\infty} \sum_{i=1}^{n-1} \pi \left(f(x_i) \right)^2 \Delta x = \int_a^b \pi \left(f(x) \right)^2 dx$$

$$\boxed{V = \pi \int_a^b \left(f(x) \right)^2 dx}$$

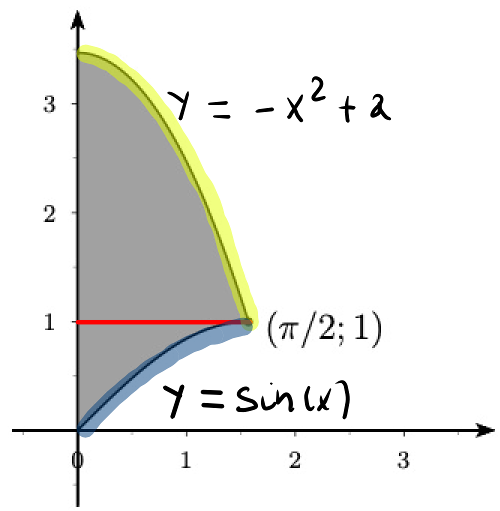
Revenons à $y = \sqrt{1-x^2}$

$$V_{\text{sphère}} = \pi \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^2 dx = \pi \int_{-1}^{1} 1 - x^2 dx$$

$$= \pi \left[x - \frac{x^3}{3} \Big|_{-1}^{1} \right] = \pi \left[\frac{3x - x^3}{3} \right]_{-1}^{1} = \pi \left(\frac{2}{3} - \frac{-2}{3} \right) = \frac{4}{3}\pi$$

2.2.28 Calculer d'abord la valeur du paramètre a, puis l'aire du domaine grisé.

a) $y = \sin(x)$, $\quad y = -x^2 + a$



$$f(x) = \sin(x) \quad , \quad f\left(\frac{\pi}{2}\right) = 1$$

$$g(x) = -x^2 + a \quad , \quad g\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow -\frac{\pi^2}{4} + a = 1 \Rightarrow a = 1 + \frac{\pi^2}{4}$$

$$\boxed{a = \frac{\pi^2 + 4}{4}}$$

Aire :

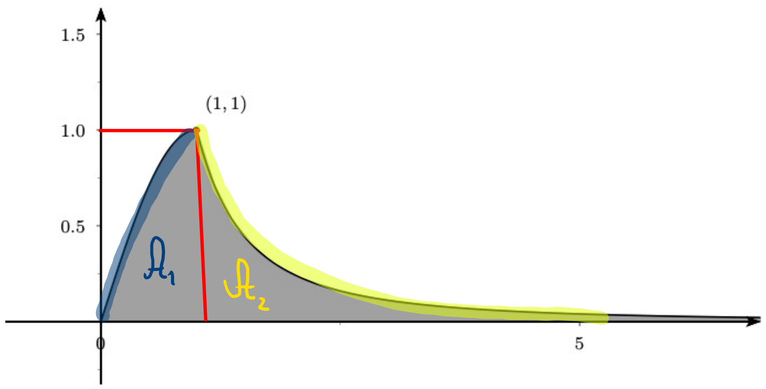
$$\int_0^{\pi/2} -x^2 + a - \sin(x) \, dx = \left[-\frac{x^3}{3} + ax + \cos(x) \right]_0^{\pi/2}$$

$$= \left(-\frac{\pi^3}{24} + \frac{\pi^2 + 4}{4} \cdot \frac{\pi}{2} + 0 \right) - 1 = \frac{-\pi^3}{24} + \frac{\pi^3 + 4\pi}{8} - 1$$

$$= \frac{-\pi^3 + 3\pi^3 + 12\pi - 24}{24} = \frac{2\pi^3 + 12\pi - 24}{24} = \frac{\pi^3 + 6\pi - 12}{12}$$

EX 2.2.28

d) $y = \sin(ax), \quad y = \frac{1}{x^2}$



$f(x) = \sin(ax)$, $f(1) = 1 \Rightarrow \sin(a) = 1 \Rightarrow a = \frac{\pi}{2}$

$g(x) = \frac{1}{x^2}$, $g(1) = 1$

$$A_1 = \int_0^1 \sin\left(\frac{\pi}{2}x\right) dx = -\frac{2}{\pi} \cos\left(\frac{\pi}{2}x\right)\Big|_0^1 = \frac{-2}{\pi}\left(0 - 1\right) = \frac{2}{\pi}$$

$$c = K \cos\left(\frac{\pi}{2}x\right)$$

$$c' = K\left(-\sin\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2}\right) = K \cdot \frac{-\pi}{2} \sin\left(\frac{\pi}{2}x\right) \Rightarrow K = \frac{-2}{\pi}$$

$$A_2 = \lim_{t \to +\infty} \left(\int_1^t \frac{1}{x^2} dx \right) = \lim_{t \to +\infty} \left(\int_1^t x^{-2} dx \right) = \lim_{t \to +\infty} \frac{-1}{x}\Big|_1^t$$

$$= \lim_{t \to +\infty} \left(\frac{-1}{t} + \frac{1}{1} \right) = 1$$

$$A = \frac{2}{\pi} + 1 = \frac{\pi + 2}{\pi}$$