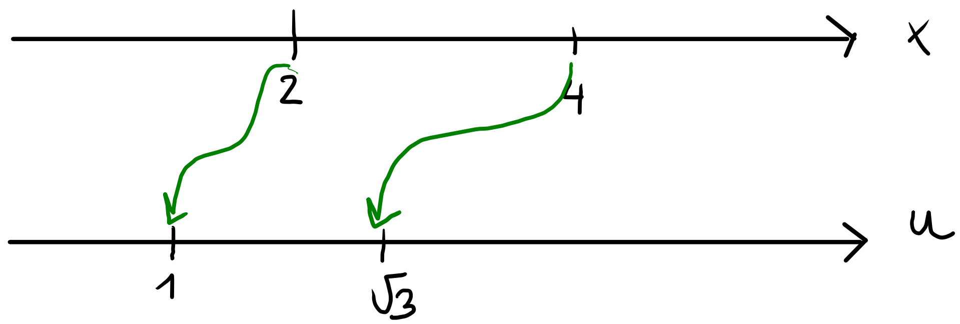
2.2.17 Calculer les intégrales suivantes à l'aide du changement de variable indiqué :

c) $\int_2^4 \frac{1}{x\sqrt{x-1}}\,dx, \quad x = u^2+1$

$f(x) = \frac{1}{x\sqrt{x-1}}$ $\qquad ED(f) =]1;+\infty[$



x	u
2	1
4	$\sqrt{3}$

changement de variable :

$x = u^2 + 1 \Leftrightarrow u^2 = x - 1 \Leftrightarrow u = \sqrt{x-1}$

$dx = 2u\,du$

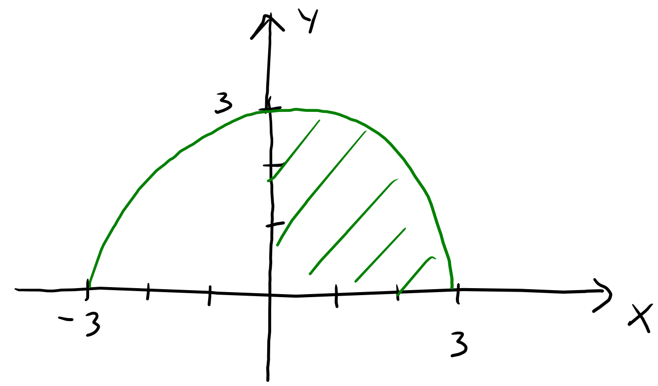
$$= \int_1^{\sqrt{3}} \frac{1}{(u^2+1)\cdot \cancel{u}} \cdot 2\cancel{u}\,du = 2\int_1^{\sqrt{3}} \frac{du}{u^2+1} \overset{\text{CRM}}{=} 2\cdot \arctan(u)\Big|_1^{\sqrt{3}} = 2\left(\arctan(\sqrt{3}) - \arctan(1)\right)$$

$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$a = 1$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

b) $\int_0^3 \sqrt{9-x^2}\,dx, \quad x = 3\sin(t)$



Géométrique : $y = \sqrt{9-x^2}$

$-3 \leq x \leq 3$

$y = \sqrt{9-x^2} \iff y^2 = 9 - x^2 \iff x^2 + y^2 = 9$ demi-cercle

$y \geq 0, \quad -3 \leq x \leq 3$

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi 3^2 = \frac{9}{4}\pi$$

Analytique :

x	t
0	0
3	$\frac{\pi}{2}$

$x = 3\sin(t) \iff \frac{x}{3} = \sin(t) \iff t = \arcsin\left(\frac{x}{3}\right)$

$dx = 3\cos(t)\,dt$

$$\int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2(t)} \cdot 3\cos(t)\,dt = \int_0^{\frac{\pi}{2}} \sqrt{9(1-\sin^2(t))} \cdot 3\cos(t)\,dt$$

$$= 9\int_0^{\frac{\pi}{2}} \sqrt{\underbrace{1-\sin^2(t)}_{\cos^2(t)}}\,\cos(t)\,dt = 9\int_0^{\frac{\pi}{2}} |\underbrace{\cos(t)}_{\geq 0}|\,\cos(t)\,dt = 9\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt$$

$0 \leq t \leq \frac{\pi}{2}$

$$\overset{\text{CRM}}{=} 9 \cdot \frac{1}{2}\left(t + \sin(t)\cos(t)\right)\Big|_0^{\frac{\pi}{2}} = \frac{9}{2}\left(\left(\frac{\pi}{2} + 0\right) - (0+0)\right) = \frac{9\pi}{4}$$

CRM

$\cos^2(x)$	$\frac{1}{2}(x + \sin(x)\cos(x))$

2.2.18 Calculer les intégrales suivantes en effectuant une intégration par parties :

b) $\displaystyle\int_0^{\pi/2} \sin(x)\cos(x)\,dx \quad = \quad \sin^2(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx$

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Par parties	$\displaystyle\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx$

$$\int u'v = uv - \int uv'$$

$u = \sin(x)$	$u' = \cos(x)\,dx$
$v = \sin(x)$	$v' = \cos(x)\,dx$

$$\Rightarrow \quad 2\int_0^{\pi/2} \sin(x)\cos(x)\,dx = \sin^2(x)\Big|_0^{\pi/2} = 1 \quad \Rightarrow \quad \int_0^{\pi/2} \sin(x)\cos(x)\,dx = \frac{1}{2}$$

d) $\int_0^3 x\sqrt{1+x}\,dx = \frac{2}{3} x \sqrt{(1+x)^3}\Big|_0^3 - \int_0^3 \frac{2}{3}(1+x)^{\frac{3}{2}}\,dx = (*)$

$$\boxed{\int f'g = fg - \int fg'}$$

$$f = \frac{2}{3}(1+x)^{\frac{3}{2}} \qquad f' = (1+x)^{\frac{1}{2}}\,dx$$

$$g = x \qquad g' = dx$$

$$(*) = \frac{2}{3}\cdot 3\left(\sqrt{1+3}\right)^3 - 0 - \frac{2}{3}\int_0^3 (1+x)^{\frac{3}{2}}\,dx$$

$$= 16 - \frac{2}{3}\cdot\frac{2}{5}(1+x)^{\frac{5}{2}}\Big|_0^3 = 16 - \frac{4}{15}\cdot(32-1) = 16 - \frac{124}{15} = \frac{240-124}{15} = \frac{116}{15}$$

2.2.19 Calculer les intégrales définies suivantes.

a) $\displaystyle\int_1^2 \frac{x}{x+6}\,dx = \int_1^2 1 + \frac{-6}{x+6}\,dx = \int_1^2 dx - 6\int_1^2 \frac{dx}{x+6}$

$$\begin{array}{r|l} x & x+6 \\ \cline{2-2} -\ \ x+6 & 1 \\ \cline{1-1} -6 & \end{array}$$

CRM

$\frac{1}{x}$	$\ln\lvert x\rvert$

$$= x\Big|_1^2 - 6\ln|x+6|\Big|_1^2 = (2-1) - 6\left(\ln(8) - \ln(7)\right)$$

$$= 1 - 6\ln\left(\frac{8}{7}\right)$$