21. Établir le développement limité d'ordre n au voisinage de a des fonctions suivantes et estimer le reste d'ordre n dans l'intervalle $[a - \frac{1}{2}\,; a + \frac{1}{2}]$.

a)	$x \mapsto \cos(x)$	$n = 5$	$a = 0$
b)	$x \mapsto e^x$	$n = 5$	$a = 0$
c)	$x \mapsto x^2 + 3x - 5$	$n = 3$	$a = 0$
d)	$x \mapsto \frac{1}{x}$	$n = 4$	$a = -2$
e)	$x \mapsto \sqrt{x}$	$n = 3$	$a = 9$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

a) $f(x) = \cos(x)$ $\qquad f(0) = 1$

$f'(x) = -\sin(x)$ $\qquad f'(0) = 0$

$f''(x) = -\cos(x)$ $\qquad f''(0) = -1$

$f^{(3)}(x) = \sin(x)$ $\qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x)$ $\qquad f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin(x)$ $\qquad f^{(5)}(x) = 0$

$P_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \; [+ R_6(x)]$ pour $f(x)$ $\qquad P_5(x)$ est paire

b) $f(x) = e^x$, $f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N}^*$

$f(0) = 1$, $f^{(n)}(0) = 1$, $\forall n \in \mathbb{N}^*$

$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + [R_6(x)]$ pour $f(x)$

c) $f(x) = x^2 + 3x - 5$, $f(0) = -5$

$f'(x) = 6x + 3$, $f'(0) = 3$

$f''(x) = 6$, $f''(0) = 6$

$f^{(n)}(x) = 0$, $n \geq 3$

$$P_3(x) = -5 + 3x + \frac{6}{6}x^2 + [R_4(x)]$$
$$= x^2 + 3x - 5 \quad = f(x)$$

d) $f(x) = \sqrt{x}$, $f(9) = 3$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f'(9) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, $f''(9) = -\frac{1}{4} \cdot \frac{1}{27} = \frac{-1}{108}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$, $f'''(9) = \frac{3}{8} \cdot \frac{1}{243} = \frac{3}{1944}$

$$P_3(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{108} \cdot \frac{1}{2}(x-)^2 + \frac{3}{1944} \cdot \frac{1}{6}(x-9)^3$$
$$= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3$$

g) $x \mapsto e^{\sin(x)}$ $\quad n = 3 \quad a = 0$

- $f(x) = e^{\sin(x)}$, $f(0) = 1$
- $f'(x) = \cos(x) \cdot e^{\sin(x)}$, $f'(0) = 1$
- $f''(x) = -\sin(x)\, e^{\sin(x)} + \cos^2(x)\, e^{\sin(x)}$, $f''(0) = 1$

$f''(x) = \left(\cos^2(x) - \sin(x)\right) e^{\sin(x)}$

- $f'''(x) = \left(-2\sin(x)\cos(x) - \cos(x)\right) e^{\sin(x)} + \cos(x)\left(\cos^2(x) - \sin(x)\right) e^{\sin(x)}$

$$= \cos(x)\, e^{\sin(x)} \left(-2\sin(x) - 1 + \cos^2(x) - \sin(x)\right)$$
$$= \cos(x)\, e^{\sin(x)} \left(\cos^2(x) - 3\sin(x) - 1\right)$$

$f'''(0) = 0$

$$P_3(x) = 1 + x + \frac{x^2}{2}$$