Exercice

$$h: \; P_3[x] \longrightarrow \mathbb{R}^3$$
$$p \longmapsto (p(-1), p(0), p(1))$$

1) $p(x^3) = (-1, 0, 1)$

$p(2x^2 - x) = (3, 0, 1)$

$p(2x^3 - 4x + 1) = (3, 1, -1)$

2) Base. $B_1 = (x^3, x^2, x, 1)$ et $B_2 = (e_1, e_2, e_3)$

$$H_{B_1}^{B_1} = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

3) Ker(h) :

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \underset{\sim}{L_1 \leftrightarrow L_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \underset{\sim}{L_3 \leftarrow L_3 + L_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$\underset{\sim}{L_3 \leftarrow \tfrac{1}{2} L_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \underset{\sim}{L_1 \leftarrow L_1 - L_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \underset{\sim}{L_3 \leftrightarrow L_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow P(ax^3 + bx^2 + cx + d) = 0 \iff \begin{cases} a = -c \\ b = d = 0 \end{cases}$$

$$\mathrm{Ker}(h) = \{ ax^3 - ax \mid a \in \mathbb{R} \} = \langle (1, 0, -1, 0) \rangle$$

Comme $\dim(\mathrm{Ker}(h)) = 1$, on a $\dim(\mathrm{Im}(h)) = 3$

Donc $\mathrm{Im}(h) = \mathbb{R}^3$