2.2.16 Déterminer la nature des extremums des fonctions suivantes :

a) $f : x \longmapsto \int_0^x (t^3 - t)\, dt$

$$f(x) = \int_0^x (t^3 - t)\, dt = \frac{1}{4}x^4 - \frac{1}{2}x^2 \qquad x \in \mathbb{R}$$

$$f'(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$$

x		-1		0		1	
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$

max $(0, 0)$

min $\left(-1 ; -\frac{1}{4}\right)$

$\left(1 ; -\frac{1}{4}\right)$

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d) $\int_k^0 \frac{2}{(x+1)^3}\,dx = -\int_0^k \frac{3}{(x+3)^2}\,dx$.

$$\int_K^0 2(x+1)^{-3}\,dx = -(x+1)^{-2}\Big|_K^0 = -\left[1^{-2} - (K+1)^{-2}\right]$$

$$= -1 + \frac{1}{(K+1)^2}$$

$$-\int_0^K 3(x+3)^{-2}\,dx = -\left[-3(x+3)^{-1}\right]_0^K = 3\frac{1}{x+3}\Big|_0^K$$

$$= 3\frac{1}{K+3} - 3\frac{1}{3}$$

$$= \frac{3}{K+3} - 1$$

On résout l'équation :

$$-1 + \frac{1}{(K+1)^2} = \frac{3}{K+3} - 1$$

$$\frac{1}{(K+1)^2} - \frac{3}{K+3} = 0 \qquad K \neq -1 \text{ et } K \neq -3$$

$$(K+3) - 3(K+1)^2 = 0$$

$$3K^2 + 5K = 0$$

$$K(3K+5) = 0 \iff \begin{cases} K = 0 \\ \text{ou} \\ K = -\frac{5}{3} \end{cases}$$

2.2.17 Calculer les intégrales suivantes à l'aide du changement de variable indiqué :

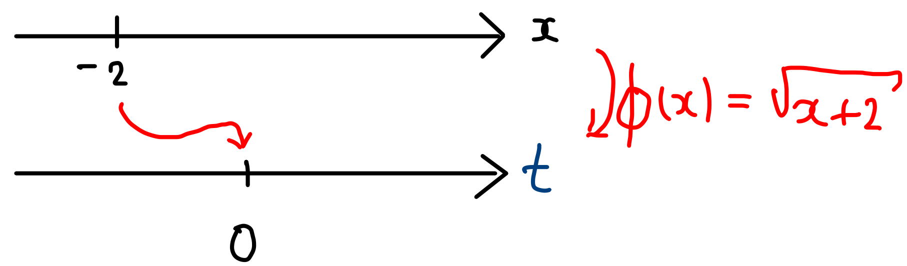
a) $\displaystyle\int_{-2}^{0} x\sqrt{x+2}\,dx, \quad x = t^2 - 2$

$f(x) = x\sqrt{x+2} \qquad \mathcal{D}(f) = [-2\,;+\infty[$

$x = t^2 - 2 \quad ; \quad t = \sqrt{x+2} \quad , \quad t \in \mathbb{R}_+$

$dx = 2t\,dt$

x	t
-2	0
0	$\sqrt{2}$



$$\int_{-2}^{0} x\sqrt{x+2}\,dx = \int_{0}^{\sqrt{2}} (t^2-2)\,t\,2t\,dt = 2\int_{0}^{\sqrt{2}} t^2(t^2-2)\,dt$$

$$= 2\int_{0}^{\sqrt{2}} t^4 - 2t^2\,dt = 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right] = 2\,\frac{(12-20)\sqrt{2}}{15} = \frac{-16}{15}\sqrt{2}$$

Intégration par parties

2.2.18 Calculer les intégrales suivantes en effectuant une intégration par parties :

a) $\displaystyle\int_0^{\pi} x\sin(x)\,dx$ c) $\displaystyle\int_0^{\pi/4} \cos^2(x)\,dx$

$$(u\cdot v)' = u'v + uv'$$

$$u'v = (uv)' - uv'$$

$$\int u'v = \int (uv)' - \int uv'$$

$$\int u'v = uv - \int uv'$$

a) $$\int x\sin(x)\,dx = -x\cos(x) - \int(-\cos(x))\,dx$$

$u' = \sin(x)\,dx$ $u = -\cos(x)$

$v = x$ $v' = dx$

$$= -x\cos(x) + \int \cos(x)\,dx$$

$$= -x\cos(x) + \sin(x) + C$$

2.2.17

b) $\int_0^3 \sqrt{9-x^2}\,dx, \quad x = 3\sin(t)$

$f(x) = \sqrt{9-x^2} \qquad D(f) = [-3;3]$

$x = 3\sin(t) \quad ; \quad t = \arcsin\left(\frac{x}{3}\right)$

$dx = 3\cos(t)\,dt \quad ;$

x	t
0	0
3	$\frac{\pi}{2}$

$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$$\int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\underbrace{9-9\sin^2(t)}_{9\cos^2(t)}} \cdot 3\cos(t)\,dt$$

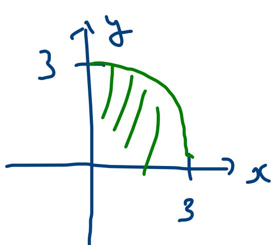
$$= \int_0^{\pi/2} 3\underbrace{|\cos(t)|}_{>0} \cdot 3\cos(t)\,dt = 9\int_0^{\pi/2} \cos^2(t)\,dt$$

$$= 9\int_0^{\pi/2} \frac{(1+\cos(2t))}{2}\,dt = \frac{9}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9}{2}\left[\left(\frac{\pi}{2}+0\right) - (0-0)\right] = \frac{9\pi}{4}$$

Géométriquement : $y = \sqrt{9-x^2} \qquad 0 \leq x \leq 3$

$y^2 = 9 - x^2 \implies x^2 + y^2 = 9$



$\frac{1}{4}$ cercle : $\frac{1}{4}\pi \cdot 9 = \frac{9}{4}\pi$