4.1

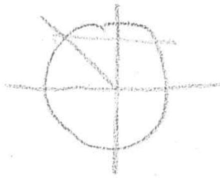
$$y = a\cos(\omega_0 t) + b\sin(\omega_0 t) \Rightarrow y = A\cos(\omega_0 t - \delta)$$

où $A = \sqrt{a^2+b^2}$, $\cos(\delta) = \frac{a}{A}$ et $\sin(\delta) = \frac{b}{A}$

a) $3\cos(2t) + 4\sin(2t)$

$A = 5$ et $\begin{cases} \cos(\delta) = \frac{3}{5} \\ \sin(\delta) = \frac{4}{5} \end{cases} \Rightarrow \delta = 0{,}92730$

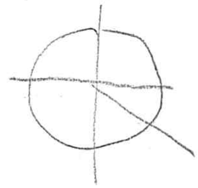
$\Rightarrow \quad 5\cos(2t - 0{,}92730)$



b) $-\cos(t) + \sqrt{3}\sin(t)$

$A = 2$ et $\begin{cases} \cos(\delta) = \frac{-1}{2} \\ \sin(\delta) = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \delta = 120° = \frac{2\pi}{3}$

$\Rightarrow \quad 2\cos\left(t - \frac{2\pi}{3}\right)$

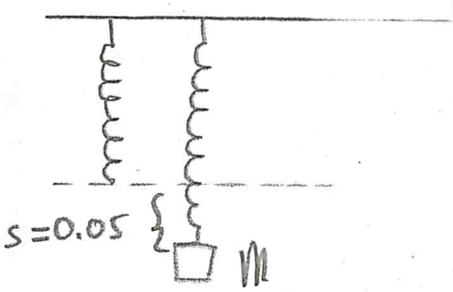


c) $4\cos(3t) - 2\sin(3t)$

$A = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$ et $\begin{cases} \cos(\delta) = \frac{2}{\sqrt{5}} \\ \sin(\delta) = -\frac{1}{\sqrt{5}} \end{cases} \Rightarrow \delta = -0{,}46$

$\Rightarrow \quad 2\sqrt{5}\cos(3t + 0{,}46365)$

4.2



$m = 100g = 0{,}1\ Kg$

$v = 10\ cm/s = 0{,}1\ m/s = y'(0)$

$y(0) = 0$

1) Déterminons K: $mg = Ks$

$$0{,}1 \cdot 9{,}8 = K \cdot 0{,}05$$

$$\Rightarrow K = \frac{0{,}98}{0{,}05} = 19{,}6$$

2) Déterminons ω_0 : $\omega_0^2 = \frac{K}{m} = \frac{19{,}6}{0{,}1} = 196 \Rightarrow \omega_0 = 14$

3) $y'' + 196y = 0$ avec $y(0) = 0$ et $y'(0) = 0{,}1$

$$r^2 + 196 = 0 \quad \Rightarrow r = \pm 14i$$

$$\boxed{y = a\cos(14t) + b\sin(14t)}$$

$$y' = -\underbrace{14a\sin(14t)}_{} + 14b\underbrace{\cos(14t)}_{}$$

$$\Rightarrow y'(0) = 0 + 14b \cdot 1 = 0{,}1 \Rightarrow \boxed{b = \frac{0{,}1}{14} = \frac{1}{140}}$$

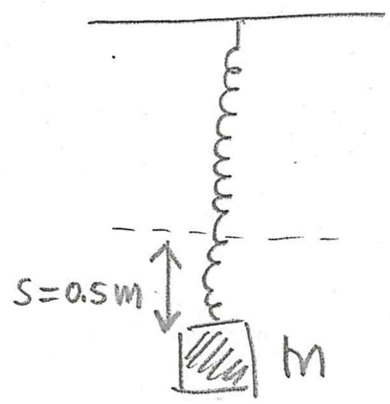
$$\Rightarrow y(0) = 0 \Rightarrow a \cdot 1 + b \cdot 0 = 0 \Rightarrow \boxed{a = 0}$$

Finalement $\boxed{y = \frac{1}{140}\sin(14t)}$

4.3

$K = 700 \ N/m$

$m = 7 \ Kg$



1) $y(0) = 0{,}5$ <u>et</u> $y'(0) = 0$

$$\omega_0^2 = \frac{700}{7} = 100 \Rightarrow \omega_0 = 10$$

Ainsi l'ED : $y'' + 100y = 0 \quad (r = \pm 10i)$

éq. du mvt : $\boxed{y = a\cos(10t) + b\sin(10t)}$

$$y' = -10a\sin(10t) + 10b\cos(10t)$$

- $y'(0) = 0 \Rightarrow 0 = b$
- $y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = a$

Finalement : $y = \frac{1}{2}\cos(10t)$

2) C'est un système amorti.

On a l'équation

$$7y'' + \frac{1}{4}y' + 700y = 0 \qquad (4.6 \text{ page } 53)$$

(4.3 suite) ②

Résolvons cette équation :

$$y'' + \frac{1}{28}y' + 100 = 0$$

$$\Delta = \frac{-313599}{784} \quad \text{de } r^2 + \frac{1}{28}r + 100 = 0$$

Les solutions sont complexes : $r = \frac{-1}{56} \pm i\frac{\sqrt{313599}}{56}$

ou : $r \cong -0{,}017857 \pm 10i$

La solution : $y \cong e^{-0{,}017857t}\left(C_1\cos(10t) + C_2\sin(10t)\right)$

- $y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = C_1$

- $y' = \lambda e^{\lambda t}\left(\frac{1}{2}\cos(10t) + C_2\sin(10t)\right)$

$$+ e^{\lambda t}\left(-5\sin(10t) + 10C_2\cos(10t)\right)$$

$$y'(0) = \lambda\left(\frac{1}{2}\right) + 10C_2$$

Donc $\frac{\lambda}{2} + 10C_2 = 0$

$$C_2 = -\frac{\lambda}{20} \cong \frac{1}{1120} \cong 9 \cdot 10^{-4} \cong 0$$

$$\boxed{y = \frac{1}{2}e^{-0{,}01786t}\cos(10t)}$$

$$7y'' + 980y' + 700y = 0 \qquad (4.6 \text{ page } 53)$$

$$y'' + 140y' + 100y = 0$$

résolvons $r^2 + 140r + 100 = 0$

$$r = -70 \pm 40\sqrt{3} \qquad \text{avec } \Delta = 19'200 = (80\sqrt{3})^2$$

$$r_1 \cong -139{,}28203 \text{ et } r_2 \cong -0{,}71797$$

La solution: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

- $y(0) = \frac{1}{2} \Rightarrow c_1 + c_2 = \frac{1}{2}$

$$y' = c_1 r_1 e^{r_1 t} + c_2 c_2 e^{r_2 t}$$

- $y'(0) = 0 \Rightarrow c_1 r_1 + c_2 r_2 = 0$

Résolvons le système:

$$\begin{cases} c_1 + c_2 = \frac{1}{2} \\ r_1 c_1 + r_2 c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_2 = \frac{1}{2} - c_1 \\ r_1 c_1 + r_2\left(\frac{1}{2} - c_1\right) = 0 \quad (*) \end{cases}$$

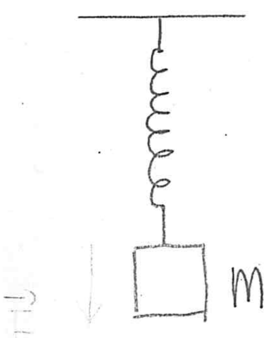
$(*)$ $r_1 c_1 + \frac{1}{2} r_2 - r_2 c_1 = 0$

$$c_1 (r_1 - r_2) = -\frac{1}{2} r_2 \Rightarrow c_1 = \frac{-r_2}{2(r_1 - r_2)} \cong -0{,}00259$$

et $c_2 \cong 0{,}50260$

$$y = -0{,}00259 e^{-139{,}28203t} + 0{,}50260 e^{-0{,}71797t}$$

4.4 Vibrations libres amorties



$m = 2\,[Kg]$

1) Déterminons K: $3 = K \cdot 0{,}1 \Rightarrow K = 30\ [N/m]$

2) Si $y' = 5\ [m/s]$ alors $cy' = 3\ [N]$

Donc $c = \frac{3}{5}\ [Ns/m]$

3) Equation du mouvement:

$$2y'' + \frac{3}{5}y' + 30y = 0$$

avec $y(0) = 0{,}05\,[m]$ et $y'(0) = 0{,}1\ [m/s]$

$$2r^2 + \frac{3}{5}r + 30 = 0 \qquad \Delta < 0$$

donne $r = \dfrac{-3 \pm \sqrt{5991}\,i}{20} \cong -0{,}15 \pm 3{,}87008\,i$

système faiblement amorti $(\Delta < 0)$

Par (4.8): $u = \dfrac{\sqrt{4 \cdot 30 \cdot 2 - \left(\frac{3}{5}\right)^2}}{2 \cdot 2} \cong 3{,}87008$

Par (4.9): $y = Ae^{\frac{-\frac{3}{5}t}{4}} \cos(3{,}87008t - \delta)$

$$y = Ae^{-0{,}15t} \cos(3{,}87008t - \delta)$$

4.4 (suite)

Déterminons A et δ.

- $y(0) = 0{,}05 \Rightarrow 0{,}05 = A\cos(-\delta) = A\cos(\delta)$

$$y' = -0{,}15Ae^{-0{,}15t}\cos(3{,}87008t - \delta)$$
$$- 3{,}87008Ae^{-0{,}15t}\sin(3{,}87008t - \delta)$$

$$= -Ae^{-0{,}15t}\left[0{,}15\cos(\lambda t - \delta) + \lambda\sin(\lambda t - \delta)\right]$$

- $y'(0) = -A\left(0{,}15\cos(-\delta) + \lambda\sin(-\delta)\right)$
$$= -A\left(-0{,}15\cos(\delta) - \lambda\sin(\delta)\right)$$
$$= -A\left(0{,}15\cos(\delta) - \lambda\sin(\delta)\right)$$

Comme $A = \dfrac{0{,}05}{\cos(\delta)}$, on a :

$$-\frac{0{,}05}{\cos(\delta)}\left(0{,}15\cos(\delta) - \lambda\sin(\delta)\right) = 0{,}1$$
$$-0{,}0075 + 0{,}05\lambda\tan(\delta) = 0{,}1$$

$$\tan(\delta) = \frac{0{,}1 + 0{,}0075}{0{,}05\cdot\lambda} \cong 0{,}55554$$

Donc $\delta \cong 0{,}50790$ et $A = 0{,}05720$

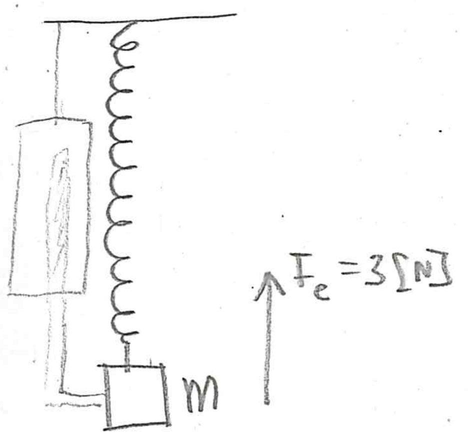
$$\boxed{y = 0{,}05720\, e^{-0{,}15t}(3{,}87008t - 0{,}50709)}$$

4.5

$K = 13 \, [kg/s^2]$

$m = 0{,}25 \, kg$

$F_{am} = 2y'$



Equation du mouvement:

$$0{,}25\, y'' + 2y' + 13y = -3$$

Qui s'écrit :

$$y'' + 8y' + 52y = -12$$

1) Equation sans second membre.

$r^2 + 8r + 52 = 0 \qquad \Delta = -144 = 144 i^2$

$r = -4 \pm 6i$

Solution : $\boxed{y = e^{-4t}\left(c_1 \cos(6t) + c_2 \sin(6t)\right)}$

2) Solution particulière.

$y = at + b \; ; \; y' = a, \; y'' = 0$

$0 + 8a + 52at + 52b = -12$

Donc $a = 0$ et $b = \frac{-12}{52} = \frac{-3}{13}$

Solution : $\boxed{y = e^{-4t}\left(c_1 \cos(6t) + c_2 \sin(6t)\right) - \frac{3}{13}}$

(4.5 suite)

3) Déterminons c_1 et c_2.

Conditions initiales : $y(0) = 0,15$ et $y'(0) = 0$

• $y(0) = 0,15$:

$$c_1 - \frac{3}{13} = 0,15$$

$$c_1 = 0,15 + \frac{3}{13} = \frac{3}{20} + \frac{3}{13} = \frac{99}{260}$$

• $y'(0) = 0$

$$y' = -4e^{-4t}\left(c_1 \cos(6t) + c_2 \sin(6t)\right)$$
$$+ e^{-4t}\left(-6c_1 \sin(6t) + 6c_2 \cos(6t)\right)$$

$$y'(0) = -4c_1 + 6c_2 = 0$$

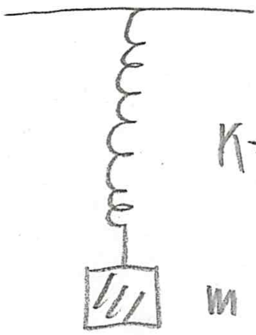
$$\Rightarrow \quad c_2 = \frac{4c_1}{6} = \frac{33}{130}$$

Finalement :

$$\boxed{y = e^{-4t}\left(\frac{99}{260}\cos(6t) + \frac{33}{130}\sin(6t)\right) - \frac{3}{13}}$$

4.6

Equation du mouvement:



$K = 4\,[N/m]$

$m = 1\,[kg]$

$$y'' + 4y = 1 + t + \sin(2t)$$

1) Eq. homogène: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$y = C_1 \cos(2t) + C_2 \sin(2t)$$

⚠ $p\cos(2t) + q\sin(2t)$

2) Sol. particulière

$$y = \underbrace{a + bt}_{y_1} + \underbrace{t(A\cos(2t) + B\sin(2t))}_{y_2}$$

• $y_1' = b$

$y_1'' = 0 \Rightarrow 4a + 4bt = 1 + t \Rightarrow \begin{cases} a = \frac{1}{4} \\ b = \frac{1}{4} \end{cases}$

• $y_2' = A\cos(2t) + B\sin(2t) + t\left(-2A\sin(2t) + 2B\cos(2t)\right)$

$$y_2'' = \underline{-2A\sin(2t)} + \underline{2B\cos(2t)} \underline{- 2A\sin(2t)} \underline{+ 2B\cos(2t)} + t\left(-4A\cos(2t) - \underline{4B\sin(2t)}\right)$$

$$= -4A\sin(2t) - 4Bt\sin(2t) + 4B\cos(2t) - 4At\cos(2t)$$

$$= (-4A - 4Bt)\sin(2t) + (4B - 4At)\cos(2t)$$

$$y'' + 4y = (-4A - 4Bt + 4Bt)\sin(2t) + (4B - 4At + 4At)\cos(2t)$$

$$= -4A\sin(2t) + 4B\cos(2t)$$

donc $A = -\frac{1}{4}$ et $B = 0$

(4.6 suite)

Solution

$$\boxed{y = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4}t + \frac{1}{4} - \frac{1}{4}t\cos(2t)}$$

Déterminons C_1 et C_2 :

- $y(0) = 0$ et $y'(0) = 0$

$$y(0) = C_1 + \frac{1}{4} = 0 \quad \Rightarrow \quad C_1 = -\frac{1}{4}$$

- $y' = -2C_1\sin(2t) + 2C_2\cos(2t) + \frac{1}{4} - \frac{1}{4}\cos(2t) - \frac{1}{2}t\sin(2t)$

$$y'(0) = 2C_2 + \frac{1}{4} - \frac{1}{4} \quad \Rightarrow \quad C_2 = 0$$

$$y = -\frac{1}{4}\cos(2t) + \frac{1}{4}t + \frac{1}{4} - \frac{1}{4}t\cos(2t)$$

$$= \frac{1}{4}\cos(2t)(-1-t) + \frac{1}{4}(t+1)$$

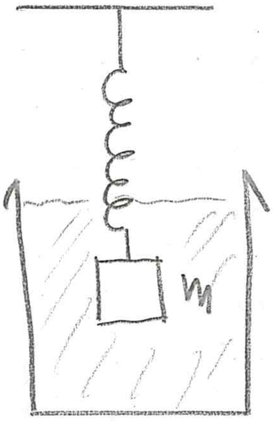
$$= \frac{1}{4}(t+1)\Big[\underbrace{-\cos(2t) + 1}_{2\sin^2(t)}\Big]$$

$\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$

$$\boxed{y = \frac{1}{2}(t+1)\sin^2(t)}$$

4.7

Système amorti



$K = 2\ [N/m]$

$F_{liq} = 4\dot{y}$

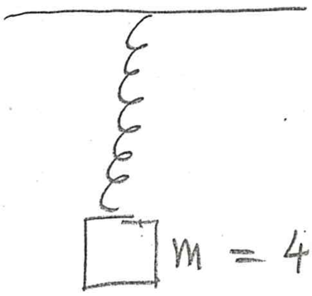
Equation du mvt : $my'' + 4y' + 2y = 0$

Résolution : $r^2 + \frac{4}{m} r + \frac{2}{m} = 0$

$$\Delta = \frac{16}{m^2} - \frac{8}{m} = \frac{8}{m}\left(\frac{2}{m} - 1\right) = \frac{8}{m}\left(\frac{2-m}{m}\right) \quad \text{avec } m > 0$$

Il faut que $\Delta > 0$, donc $0 < m < 2\ [Kg]$

4,8



$m = 4\,Kg$

1) Déterminons K : $K \cdot 0{,}125 = 4 \cdot 10 \Rightarrow K = 320$

2) Equation du mouvement :

$$4\,y'' + 320\,y = \cos(\omega t) + \sin(\omega t)$$

$$y'' + (4\sqrt{5})^2\,y = \frac{1}{4}\left[\cos(\omega t) + \sin(\omega t)\right]$$

$$y(t) = a\cos(4\sqrt{5}\,t) + b\,\sin(4\sqrt{5}\,t)$$

4.9

a) Fréquence $440 = \frac{\omega_0}{2\pi} \Rightarrow \omega_0 = 880\pi$

b) Fréquence $438 = \frac{\omega_0}{2\pi} \Rightarrow \omega_0 = 876\pi$

c) Frequence de battement : $\frac{|\omega_0 - \omega_0'|}{4\pi} = 1$ [Hz]

d) $\frac{|880\pi - \omega|}{4\pi} = 5 \Rightarrow |880\pi - \omega| = 20\pi$

$$\Rightarrow \omega = \begin{cases} 900\pi \\ 860\pi \end{cases}$$

Fréquence : $\frac{\omega}{2\pi}$ donc $\begin{cases} 450 \text{ [Hz]} \\ 430 \text{ [Hz]} \end{cases}$

4.10

a) $x'' + 4x = e^{-t}$

1) Solution éq. homogene : $y = a\cos(2t) + b\sin(2t)$

2) Solution particulière : $y_p = \lambda e^{-t}$

$$\lambda e^{-t} + 4\lambda e^{-t} = e^{-t} \Rightarrow 5\lambda = 1 \Rightarrow \lambda = \frac{1}{5}$$

$$\Rightarrow x = a\cos(2t) + b\sin(2t) + \frac{1}{5}e^{-t}$$

3) $x_0 = x_0' = 0 \Rightarrow$

$$0 = a + \frac{1}{5} \Rightarrow a = -\frac{1}{5}$$

$$0 = 2b - \frac{1}{5} \qquad b = \frac{1}{10}$$

$$x' = -2a\sin(2t) + 2b\cos(2t) - \frac{1}{5}e^{-t}$$

$$\boxed{x(t) = -\frac{1}{5}\cos(2t) + \frac{1}{10}\sin(2t) + \frac{1}{5}e^{-t}}$$

$$x'(t) = -2a\sin(2t) + 2b\cos(2t) - \frac{1}{5}e^{-t}$$

Si le terme $a\cos(2t) + b\sin(2t)$ disparaît :

b) $x(0) = a + \frac{1}{5}$ si $x(0) = \frac{1}{5} \Rightarrow a = 0$

$x'(0) = 2b - \frac{1}{5}$ si $x'(0) = -\frac{1}{5} \Rightarrow b = 0$