2.8.7 Étudier la courbure des fonctions suivantes :

a) $f(x) = 3x^2 + 8x + 10$

b) $f(x) = x^3 + 3x + 8$

c) $f(x) = (x-1)^4$

d) $f(x) = \dfrac{1}{x^2+1}$

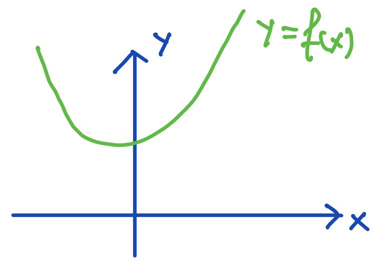
e) $f(x) = \dfrac{x}{x-1}$

f) $f(x) = \dfrac{x^3-1}{x^3+1}$

a) $ED(f) = \mathbb{R}$

$f'(x) = 6x + 8$

$f''(x) = 6$



$ED(f') = \mathbb{R}$

$ED(f'') = \mathbb{R}$

x	
$f''(x)$	$+$
$f(x)$	convexe

b) $ED(f) = \mathbb{R}$

$f'(x) = 3x^2 + 3$

$f''(x) = 6x$

$ED(f') = \mathbb{R}$

$ED(f'') = \mathbb{R}$

x		0	
$f''(x)$	$-$	0	$+$
$f(x)$		$\boxed{pi}$	

pi : $(0;8)$

c) $ED(f) = \mathbb{R}$

$f'(x) = 4(x-1)^3$

$f''(x) = 12(x-1)^2$

$ED(f') = \mathbb{R}$

$ED(f'') = \mathbb{R}$

x		1	
$f''(x)$	$+$	0	$+$
$f(x)$			

⚠ Il n'y a pas de pi en $x=1$, $f''(x)$ ne change pas de signe

d) $ED(f)=\mathbb{R}$ $\qquad\qquad \left(\frac{1}{u}\right)' = \frac{-u'}{u^2}$

$$f'(x) = \frac{-2x}{(x^2+1)^2} \qquad ED(f')=\mathbb{R}$$

<u>calcul de $f''(x)$:</u> $\quad u=-2x \;;\; u'=-2$

$v=(x^2+1)^2 \;;\; v' = 2(x^2+1)\cdot 2x$
$= 4x(x^2+1)$

$$f''(x) = \frac{-2(x^2+1)^2 - (-2x)\cdot 4x(x^2+1)}{(x^2+1)^4} = \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{2\cancel{(x^2+1)}\left[-(x^2+1)+4x^2\right]}{(x^2+1)^{\cancel{4}\,3}} = \frac{2(3x^2-1)}{(x^2+1)^3}$$

zéros de $f''(x)$: $\quad 3x^2-1=0 \iff x^2=\frac{1}{3} \iff x=\pm\frac{1}{\sqrt{3}}$

x		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
$f''(x)$	+	0	−	0	+
$f(x)$		$\boxed{\text{pi}}$		$\boxed{\text{pi}}$	

$ED(f'')=\mathbb{R}$

pi : $\left(-\frac{1}{\sqrt{3}} ; \frac{3}{4}\right)$

pi : $\left(\frac{1}{\sqrt{3}} ; \frac{3}{4}\right)$

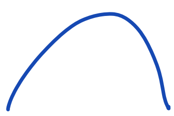
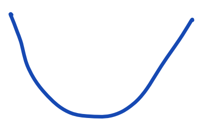
$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{\left(-\frac{1}{\sqrt{3}}\right)^2+1} = \frac{1}{\frac{1}{3}+1} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{4}$$

e) $ED(f) = \mathbb{R} - \{1\}$

$$f'(x) = \frac{1(x-1) - x \cdot 1}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2} \qquad ED(f') = \mathbb{R}-\{1\}$$

$$f''(x) = \frac{2\cancel{(x-1)}^{1}}{(x-1)^{\cancel{4}3}} = \frac{2}{(x-1)^3} \qquad ED(f'') = \mathbb{R} - \{1\}$$

x		1	
$f''(x)$	$-$	||	$+$
$f'(x)$		||	

f) $x^3 - 1 = (x-1)(x^2+x+1)$

$x^3 + 1 = (x+1)(x^2 - x + 1)$

$ED(f) = \mathbb{R} - \{-1\}$

1. $u = x^3 - 1 \quad ; \quad u' = 3x^2$

$v = x^3 + 1 \quad ; \quad v' = 3x^2$

$$f'(x) = \frac{3x^2(x^3+1) - (x^3-1)\cdot 3x^2}{(x^3+1)^2} = \frac{3x^2\left[x^3+1-x^2+1\right]}{(x^3+1)^2}$$

$$= \frac{6x^2}{(x^3+1)^2} \qquad ED(f') = \mathbb{R} - \{-1\}$$

2. $u = 6x^2 \quad ; \quad u' = 12x$

$v = (x^3+1)^2 \quad ; \quad v' = 2\cdot(x^3+1)\cdot 3x^2 = 6x^2(x^3+1)$

$$f''(x) = \frac{12x\cdot(x^3+1)^2 - 6x^2\cdot 6x^2(x^3+1)}{(x^3+1)^4} = \frac{12x(x^3+1)^2 - 36x^4(x^3+1)}{(x^3+1)^4}$$

$$= \frac{12x\cancel{(x^3+1)}\left[(x^3+1) - 3x^3\right]}{(x^3+1)^{\cancel{4}\,3}}$$

$$= \frac{12x(1-2x^3)}{(x^3+1)^3} \qquad ED(f'') = \mathbb{R} - \{-1\}$$

zéros de $f''(x)$: $12x(\underbrace{1-2x^3}) = 0$

$x = 0$ $\qquad x^3 = \frac{1}{2} \Leftrightarrow x = \frac{1}{\sqrt[3]{2}}$

Signe de $f''(x)$:

x		-1		0		$1/\sqrt[3]{2}$	
$12x$	$-$	‖	$-$	0	$+$		$+$
$1-2x^3$	$+$	‖	$+$		$+$	0	$-$
$(x^3+1)^3$	$-$	‖	$+$		$+$		$+$
$f''(x)$	$+$	‖	$-$	0	$+$	0	$-$

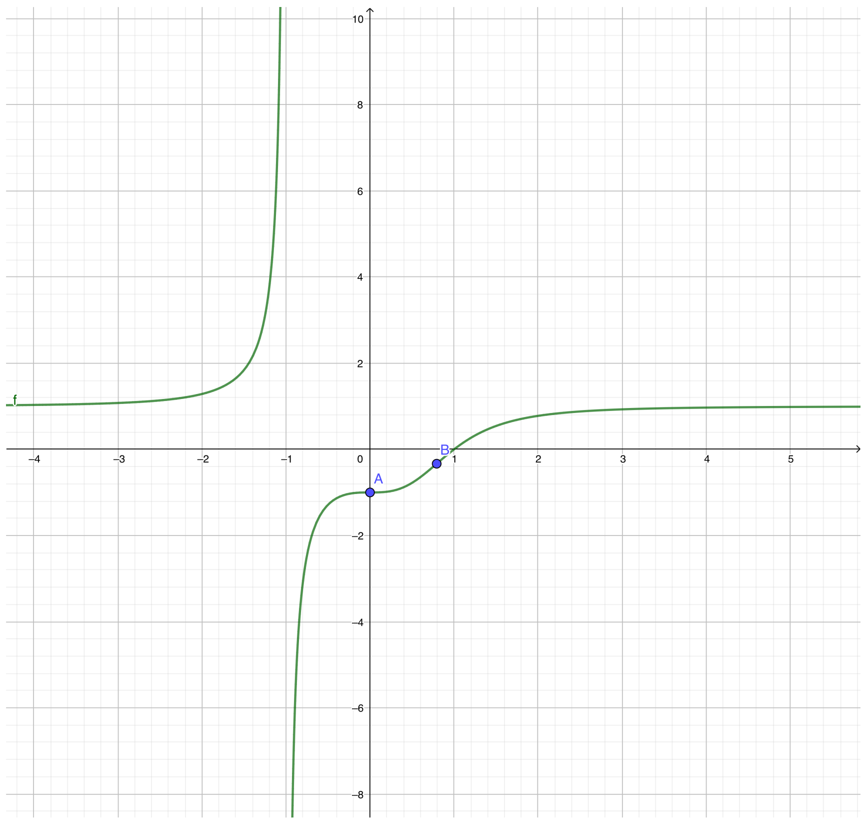
Courbure :

x		-1		0		$1/\sqrt[3]{2}$	
$f''(x)$	$+$	‖	$-$	0	$+$	0	$-$
$f(x)$		‖		pi		pi	

pi : $A(0\,;-1)$

pi : $B\left(\frac{1}{\sqrt[3]{2}}\,;-\frac{1}{3}\right)$

$$f\left(\frac{1}{\sqrt[3]{2}}\right)=\frac{\left(\frac{1}{\sqrt[3]{2}}\right)^3-1}{\left(\frac{1}{\sqrt[3]{2}}\right)+1}=\frac{\frac{1}{2}-1}{\frac{1}{2}+1}=\frac{-\frac{1}{2}}{\frac{3}{2}}=\frac{-1}{2}\cdot\frac{2}{3}=\frac{-1}{3}$$



changement de courbure en A et B.