Equations du 2ème degré dans $\mathbb{C}$

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Soit a, b et $c \in \mathbb{C}$, on résout

$$az^2 + bz + c = 0 \quad , \quad a \neq 0$$

$$az^2 + bz + c = a\left(z^2 + \frac{b}{a}z + \frac{c}{a}\right)$$

$$= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right)$$

Posons $D = \dfrac{b^2 - 4ac}{4a^2}$, il existe $d \in \mathbb{C}$ tel que $d^2 = D$

$$az^2 + bz + c = 0$$

$$\Leftrightarrow \quad a\left(z + \frac{b+d}{2a}\right)\left(z + \frac{b-d}{2a}\right) = 0$$

On déduit que toute équation complexe de degré 2 a deux solutions distinctes ou confondues.

1.3.1 Résoudre dans $\mathbb{C}$ les équations ci-dessous :

a) $z^2 = 25$

b) $z^2 = -4$

c) $2z^2 + 10z + 17 = 0$

d) $z^2 + 3z - 5 = 0$

e) $z^2 - 3(1+i)z + 6 + 7i = 0$

f) $(1+2i)z^2 - (7+4i)z + 5 - 5i = 0$

c) $2z^2 + 10z + 17 = 0$

$\Delta = 10^2 - 4 \cdot 2 \cdot 17 = 100 - 136 = -36 \quad (= D)$

Les racines carrées complexes de -36 sont $\pm 6i$.

Les solutions $z_1 = \dfrac{-\cancelto{-5}{10} + \cancelto{3}{6}i}{\cancelto{2}{4}} = \dfrac{-5+3i}{2}$, $z_2 = \overline{z_1} = \dfrac{-5-3i}{2}$

$z_1 = -\frac{5}{2} + \frac{3}{2}i$ et $z_2 = -\frac{5}{2} - \frac{3}{2}i$

e) $z^2 - 3(1+i)z + 6 + 7i = 0$

$$D = \left(-3(1+i)\right)^2 - 4 \cdot 1 \cdot (6+7i) = 9(\underbrace{1+2i+i^2}_{2i}) - 24 - 28i$$

$$= 18i - 24 - 28i = -24 - 10i$$

Déterminons d tel que $d^2 = D$. On pose $d = a + bi$

$$\begin{cases} a^2 - b^2 = -24 \\ a^2 + b^2 = \sqrt{(-24)^2 + (-10)^2} \\ 2ab = -10 \end{cases} \Leftrightarrow \begin{cases} a^2 - b^2 = -24 \\ a^2 + b^2 = 26 \\ ab = -5 \end{cases} \begin{array}{l|l} b^2 & a^2 \\ \cdot 1 & \cdot(-1) \\ \cdot 1 & \cdot 1 \\ & \end{array}$$

$$\Leftrightarrow \begin{cases} 2a^2 = 2 \\ 2b^2 = 50 \\ ab = -5 \end{cases} \Leftrightarrow \begin{cases} a = \pm 1 \\ b = \pm 5 \\ ab = -5 \end{cases} \Rightarrow d = 1 - 5i \text{ ou } d = -1 + 5i$$

On a les solutions :

$$z_1 = \frac{3(1+i) + 1 - 5i}{2} = \frac{4 - 2i}{2} = 2 - i$$

$$z_2 = \frac{3(1+i) - 1 + 5i}{2} = \frac{2 + 8i}{2} = 1 + 4i$$

$$S = \{2 - i \, ; \, 1 + 4i\}$$

f) $(1+2i)z^2-(7+4i)z+5-5i=0$

$$D = (7+4i)^2-4(1+2i)(5-5i) = 33+56i-4(15+5i) = -27+36i$$

Les racines carrées de D: $d^2 = D$ avec $d = a+bi$

$$\begin{array}{ll} \text{Re} & \\ \text{module} & \\ \text{Im} & \end{array} \left\{\begin{array}{lcl} a^2-b^2 & = & -27 \\ a^2+b^2 & = & 45 \\ 2ab & = & 36 \end{array}\right. \iff \left\{\begin{array}{l} a = \pm 3 \\ b = \pm 6 \\ ab = 18 \end{array}\right.$$

$d = 3+6i$ ou $d = -3-6i$

$$z_1 = \frac{7+4i+3+6i}{2(1+2i)} = \frac{10+10i}{2(1+2i)} = \frac{5+5i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{15-5i}{5} = 3-i$$

$$z_2 = \frac{7+4i-3-6i}{2(1+2i)} = \frac{4-2i}{2(1+2i)} = \frac{2-i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{-5i}{5} = -i$$

$S = \{3-i\,;\,-i\}$

Factorisation de polynôme

1.3.2 Décomposer dans $\mathbb{R}[z]$ et $\mathbb{C}[z]$ les polynômes ci-dessous :

a) $z^4 - 1$

d) $z^6 - 1$

$$P = z^4 - 1 = (z^2-1)(z^2+1) = \underline{(z-1)(z+1)(z^2+1)} \in \mathbb{R}[z]$$

$$P = (z-1)(z+1)(z^2-i^2) = \underline{(z-1)(z+1)(z-i)(z+i)} \in \mathbb{C}[z]$$

b) $z^4 + 1$

$$P = z^4 + 1$$

Rappel : $\omega = [r, \theta]$, l'ensemble de ses racines n-ième comporte n éléments

$$\boxed{\left\{ \left[\sqrt[n]{r} \,;\, \frac{\theta}{n} + \frac{K}{n} 2\pi \right] \middle| K = 0, \dots, n-1 \right\}}$$

Les racines 4-ième de -1 :

$$-1 = [1 ; 180°]$$

$$\omega_0 = [1 ; 45°] = \cos(45°) + \sin(45°)\, i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i = \overline{\omega_3}$$

$$\omega_1 = [1 ; 135°] = \cos(135°) + \sin(135°)\, i = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i = \overline{\omega_2}$$

$$\omega_2 = [1 ; 225°] = \cos(225°) + \sin(225°)\, i = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = \overline{\omega_1}$$

$$\omega_3 = [1 ; 315°] = \cos(315°) + \sin(315°) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = \overline{\omega_0}$$

$$P = \underline{\Big(z - \underbrace{\big(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i\big)}_{\omega_0}\Big)\Big(z - \underbrace{\big(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i\big)}_{\omega_1}\Big)\Big(z - \underbrace{\big(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i\big)}_{\omega_2}\Big)\Big(z - \underbrace{\big(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i\big)}_{\omega_3}\Big)} \in \mathbb{C}[z]$$

$$P = \boxed{(z - \omega_0)(z - \overline{\omega_0})}\boxed{(z - \omega_1)(z - \overline{\omega_1})} = (z^2 - \sqrt{2} z + 1)(z^2 + \sqrt{2} z + 1) \in \mathbb{R}[z]$$

$$\underline{\Big(z - \big(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i\big)\Big)\Big(z - \big(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i\big)\Big)} = z^2 - \Big(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i + \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i\Big) z + 1$$

$$= \underline{z^2 - \sqrt{2} z + 1}$$

$$\Big(z - \big(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i\big)\Big)\Big(z - \big(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} i\big)\Big) = \underline{z^2 + \sqrt{2} z + 1}$$

Plus rapide ...

$$P = z^4 + 1 = z^4 + 2z^2 + 1 - 2z^2$$

$$= (z^2+1)^2 - 2z^2 = (z^2 + 1 - \sqrt{2} z)(z^2 + 1 + \sqrt{2} z)$$