Ex 2.3.12 i)

$$\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)}$$

Recherche des éléments simples :

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

Donc $1 = A(x^2-x+1)+(Bx+C)(x+1)$

- $x=-1$: $1 = 3A \Rightarrow A = \frac{1}{3}$
- $x=0$: $1 = A + C \Rightarrow C = 1-\frac{1}{3} = \frac{2}{3}$
- $x=1$: $1 = A + (B+C)\cdot 2$

$1 = A + 2B + 2C \Rightarrow 1 = \frac{1}{3} + 2B + \frac{4}{3} \Rightarrow 2B = 1-\frac{5}{3}$

$\Rightarrow B = -\frac{1}{3}$

On obtient :

$$\int \frac{dx}{x^3+1} = \int \frac{1/3}{x+1}\,dx + \int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx$$

$$= I_1(x) + I_2(x)$$

$$I_1(x) = \frac{1}{3}\int \frac{dx}{x+1} = \frac{1}{3}\ln|x+1| + c$$

$$I_2(x) = -\frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx = -\frac{1}{3}\int \frac{\frac{1}{2}(2x-1)+\frac{1}{2}-2}{x^2-x+1}\,dx$$

$$= -\frac{1}{3}\cdot\frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{3}\int \frac{-\frac{3}{2}}{x^2-x+1}\,dx$$

$$= -\frac{1}{6}\ln\left(x^2-x+1\right) + \underbrace{\frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2-\frac{1}{4}+1}\,dx}_{I_3(x)}$$

$$I_3(x) = \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx \underset{a=\frac{\sqrt{3}}{2}}{\overset{\text{CRM}}{=}} \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right) + c$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

$$= \frac{1}{6}\left[2\ln|x+1| - \ln\left(x^2-x+1\right)\right] + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

$$= \frac{1}{6}\ln\left(\frac{(x+1)^2}{x^2-x+1}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c$$