Réponses

Exercice 1 (Bugnon)

L'endomorphisme : $\mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$

$p \longmapsto p + (3x-1)\cdot p' - (2x^2-5)\cdot p''$

a) $f(e_1) = f(x^2) = x^2 + (3x-1)\cdot 2x - (2x^2-5)\cdot 2 = x^2+6x^2-2x-4x^2+10$

$= 3x^2-2x+10 = \begin{pmatrix}3\\-2\\10\end{pmatrix}_B$

$f(e_2) = f(x) = x + (3x-1)\cdot 1 = 4x-1 = \begin{pmatrix}0\\4\\-1\end{pmatrix}$

$f(e_3) = f(1) = 1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$M = \begin{pmatrix}3 & 0 & 0\\ -2 & 4 & 0\\ 10 & -1 & 1\end{pmatrix}$

b) $C_M(t) = \begin{vmatrix}3-t & 0 & 0\\ -2 & 4-t & 0\\ 10 & -1 & 1-t\end{vmatrix} = (1-t)\begin{vmatrix}3-t & 0\\ -2 & 4-t\end{vmatrix} = (1-t)(3-t)(4-t)$

$= -(t-1)(t-3)(t-4)$ [la matrice est triangulaire inférieure]

$C_M(t)$ a 3 zéros distincts et est de degré 3.

f admet 3 valeurs propres distinctes, comme $\mathbb{R}_2[x]$ est de dim 3, alors f est diagonalisable. On a déjà :

$M' = \begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{pmatrix}$

E_1: $MP = P$ $\begin{cases}3x = x\\ -2x+4y = y\\ 10x-y+z = z\end{cases} \Rightarrow \begin{cases}x=y=0\\ z=z\end{cases}$

$E_1 = \{(0,0,z) \mid z\in\mathbb{R}\} = \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle$

E_3: $MP = 3P$ $\begin{cases}3x = 3x\\ -2x+4y = 3y\\ 10x-y+z = 3z\end{cases} \Rightarrow \begin{cases}x=x\\ y=2x\\ 2z = 10x-y = 8x\end{cases}$

$\Rightarrow \begin{cases}y=2x\\ z=4x\end{cases}$

$\Rightarrow E_3 = \{(x,2x,4x) \mid x\in\mathbb{R}\} = \left\langle \begin{pmatrix}1\\2\\4\end{pmatrix}\right\rangle$

$\underline{E_4}$: $MP = 4P$ $\quad \begin{cases} 3x = 4x \\ -2x + 4y = 4y \\ 10x - y + z = 4z \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = y \\ 3z = -y \end{cases}$

$\Rightarrow E_4 = \left\{ (0,\ 3z, -z) \mid z \in \mathbb{R} \right\} = \left< \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right>$

Finalement $\quad B' = \big((0,0,1),\ (1,2,4),\ (0,3,-1) \big)$ et $Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 4 & -1 \end{pmatrix}$

$$\boxed{\begin{tikzcd}
(\mathbb{R}_2[x], B) \arrow[r, "M"] & (\mathbb{R}_2[x], B) \arrow[d, "Q^{-1}"] \\
(\mathbb{R}_2[x], B') \arrow[u, "Q"] \arrow[r, "M'"] & (\mathbb{R}_2[x], B')
\end{tikzcd}}$$

$M' = Q^{-1} M Q \quad$ et $\quad M = Q M' Q^{-1}$

c) Pour calculer M^n on utilise $M = QM'Q^{-1}$

car $\underline{\underline{M^n = Q \cdot M'^n \cdot Q^{-1}}}$. $\quad M'^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$

$Q^{-1} = \frac{1}{3} \begin{bmatrix} -14 & 1 & 3 \\ 3 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

$$M^n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix} \left[\frac{1}{3} \begin{pmatrix} -14 & 1 & 3 \\ 3 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix} \right] =$$

$$\frac{1}{3} \begin{pmatrix} 0 & 3^n & 0 \\ 0 & 2 \cdot 3^n & 3 \cdot 4^n \\ 1 & 4 \cdot 3^n & -4^n \end{pmatrix} \begin{pmatrix} -14 & 1 & 3 \\ 3 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3^{n+1} & 0 & 0 \\ 2 \cdot 3^{n+1} - 6 \cdot 4^n & 3 \cdot 4^n & 0 \\ -14 + 4 \cdot 3^{n+1} + 2 \cdot \underbrace{4^n}_{2^{2n}} & 1 - 4^n & 3 \end{pmatrix}$$

$$\Rightarrow M^n = \begin{pmatrix} 3^n & 0 & 0 \\ 2 \cdot 3^n - 2 \cdot 4^n & 4^n & 0 \\ 4 \cdot 3^n + 2^{2n+1} - \frac{14}{3} & \frac{1}{3} - \frac{4^n}{3} & 1 \end{pmatrix}$$

$\ldots \quad M^5 = \begin{pmatrix} 243 & 0 & 0 \\ -1562 & 1024 & 0 \\ 1650 & -341 & 1 \end{pmatrix} \qquad f(x^2 - x - 1) = M^5 \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 243 \\ -2586 \\ 1990 \end{pmatrix}$

d) f est un endomorphisme, c'est dit. 3

Comme $|M| = 12$, f est bijective, donc f est un automorphisme

$$M^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ -38 & 3 & 12 \end{bmatrix}$$

$$\mathbb{R}_2[x] \xrightarrow{M} \mathbb{R}_2[x]$$

$$\mathbb{R}_2[x] \xleftarrow{M^{-1}} \mathbb{R}_2[x]$$

$$\longleftarrow 6x^2+12x+13$$

$$\frac{1}{12}\begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ -38 & 3 & 12 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \\ 13 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 24 \\ 48 \\ -36 \end{bmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = 2x^2+4x-3$$

<u>contrôle</u>: $f\left(2x^2+4x-3\right) = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 0 \\ 10 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ 13 \end{pmatrix}$

d)

$$t_1 = 0 : \quad V_0 = Ker(h) = \left\langle \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

$$t_2 = 1 : \begin{pmatrix} 5/9-1 & -4/9 & -2/9 \\ -4/9 & 5/9-1 & -2/9 \\ -2/9 & -2/9 & 8/9-1 \end{pmatrix} \sim \begin{pmatrix} -4/9 & -4/9 & -2/9 \\ -4/9 & -4/9 & -2/9 \\ -2/9 & -2/9 & -1/9 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$V_1 = \{(t+t' ; -t ; -2t') / t,t' \in \mathbb{R}\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\rangle$. Donc h est diagonalisable

e)

$$B^* = \left(\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right) \text{ et } D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercice

a)

$$|H| = \left(\frac{1}{9}\right)^3 \begin{vmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{vmatrix} = \left(\frac{1}{9}\right)^3 \begin{vmatrix} 9 & -4 & -18 \\ -9 & 5 & 18 \\ 0 & -2 & 0 \end{vmatrix} = \left(\frac{1}{9}\right)^3 \cdot 9 \cdot 2 \cdot 18 \cdot \begin{vmatrix} 1 & -4 & -1 \\ -1 & 5 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \frac{4}{9} \cdot (1-1) = 0$$

Variante : on voit que $2c_1 + 2c_2 + c_3 = 0$, donc $|H| = 0$

b)

$$\begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 5 & -4 & -2 \\ -4 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 0 & -9 & 18 \\ 0 & 9 & -18 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ker(h) = \{(2t; 2t; t) / t \in \mathbb{R}\} = \left\langle \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle \quad \text{Base : } \left(\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right)$$

$$\text{Im}(h) = \left\langle \begin{pmatrix} 5 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -2 \end{pmatrix} \right\rangle \quad \text{Base : } \boxed{\left(\begin{pmatrix} 5 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -2 \end{pmatrix} \right)}$$

c)

$$\begin{vmatrix} 5/9-t & -4/9 & -2/9 \\ -4/9 & 5/9-t & -2/9 \\ -2/9 & -2/9 & 8/9-t \end{vmatrix} = \left(\frac{1}{9}\right)^3 \begin{vmatrix} 5-9t & -4 & -2 \\ -4 & 5-9t & -2 \\ -2 & -2 & 8-9t \end{vmatrix} =$$

$$\left(\frac{1}{9}\right)^3 \begin{vmatrix} 9-9t & -9+9t & 0 \\ 0 & 9-9t & -18+18t \\ -2 & -2 & 8-9t \end{vmatrix} = \left(\frac{1}{9}\right)^3 (9-9t)^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ -2 & -2 & 8-9t \end{vmatrix} =$$

$$\frac{1}{9}(1-t)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ -2 & -4 & 8-9t \end{vmatrix} = \frac{1}{9}(t-1)^2 \left[1 \cdot (8-9t) - 8\right] = -t \cdot (t-1)^2$$

Les valeurs propres sont 0 et 1 (double) d)

$t_1 = 0$: $V_0 = Ker(h) = \left\langle \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

$t_2 = 1$: $\begin{pmatrix} 5/9-1 & -4/9 & -2/9 \\ -4/9 & 5/9-1 & -2/9 \\ -2/9 & -2/9 & 8/9-1 \end{pmatrix} \sim \begin{pmatrix} -4/9 & -4/9 & -2/9 \\ -4/9 & -4/9 & -2/9 \\ -2/9 & -2/9 & -1/9 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$V_1 = \{(t+t' \,;\, -t \,;\, -2t') \,/\, t,t' \in \mathbb{R}\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\rangle$. Donc h est diagonalisable

e)

$B^* = \left(\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right)$ et $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$