2.1 (66) I

a) $\frac{dy}{\cos(x)} - 2y\,dx = 0$

$dy = 2y\cos(x)\,dx$

$\frac{dy}{2y} = \cos(x)\,dx$

En intégrant : $\frac{1}{2}\int \frac{dy}{y} = \int \cos(x)\,dx$

$\frac{1}{2}\ln|y| = \sin(x) + c$

$\ln|y| = 2\sin(x) + c'$

$y = e^{2\sin(x) + c'}$

$\underline{\underline{y = K e^{2\sin(x)}}}$

b) $x\,dy = y\,dx$

$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + c$

$\underline{\underline{y = Kx}}$

c) $3y\,dx + (xy + 5x)\,dy = 0$

$(xy + 5x)\,dy = -3y\,dx \quad | \div xy$

$\left(1 + \frac{5}{y}\right)dy = -\frac{3\,dx}{x}$

$y + 5\ln|y| = -3\ln|x| + c$

$-y = \ln|y^5| + \ln|x^3| + \ln|K|$

$\underline{\underline{e^{-y} = Kx^3y^5}}$

d) $y' = x - 1 + xy - y$

$y' = (x-1) + y(x-1)$

$y' = (x-1)(y+1)$

$$\frac{y'}{y+1} = x - 1$$

$\ln|y+1| = \frac{1}{2}x^2 - x + C$

$y+1 = e^{1/2 x^2 - x + C}$

$y = Ke^{\frac{1}{2}x^2 - x} - 1$

e) $e^{x+2}\,dx = e^{2x} \cdot e^{-y}\,dy \qquad | \div e^{2x}$

$e^{-x+2}\,dx = e^{-y}\,dy$

$-e^{-x+2} + c = -e^{-y}$

$e^{-x+2} + C = e^{-y}$

$-y = \ln(e^{-x+2} + c) \Rightarrow \underline{\underline{y = -\ln(e^{-x+2} + c)}}$

2.1 (66) II

f) $y(1+x^3)y' + x^2(1+y^2) = 0$

$(1+x^3)\, yy' = -(1+y^2)x^2$

$$\frac{yy'}{1+y^2} = \frac{-x^2}{1+x^3}$$

$$\frac{1}{2}\ln|1+y^2| = -\frac{1}{3}\ln|1+x^3| + c \quad | \cdot 6$$

$$\ln|(1+y^2)^3| = -\ln|(1+x^3)^2| + c'$$

$$(1+y^2)^3 = e^{-\ln|1+x^3|^2 + c'}$$

$$(1+y^2)^3 = K\left((1+x^3)^2\right)^{-1}$$

$$(1+y^2)^3 \cdot (1+x^3)^2 = K$$

g) $e^y \sin(x)\, dx - \cos^2(x)\, dy = 0$

$e^y \sin(x)\, dx = \cos^2(x)\, dy$

$$\frac{\sin(x)}{\cos^2(x)}\, dx = \frac{dy}{e^y}$$

$$\cos^{-1}(x) + c = -e^{-y}$$

$$e^{-y} = \frac{-1}{\cos(x)} + c'$$

$$-y = \ln\left(\frac{-1}{\cos(x)} + c'\right)$$

2.2 (66)

a) $x\,dx = -y e^{-x} dy$, $y(0) = 1$

$x e^{x} dx = -y\,dy$

$\int x e^{x} dx = -\frac{1}{2}y^2 + C$

$u = e^{x}$ $\quad u' = e^{x} dx$

$v = x$ $\quad v' = dx$

$\int x e^{x} dx = e^{x} x - \int e^{x} dx = (x-1)e^{x}$

Ainsi $-\frac{1}{2}y^2 = (x-1)e^{x} + C$

$y^2 = 2(x-1)e^{x} + c'$

$y = \sqrt{2(1-x)e^{x} + c'}$

$y(0) = \sqrt{2 + c'} = 1 \Rightarrow c' = -1$

$$\underline{\underline{y = \sqrt{2(1-x)e^{x} - 1}}}$$

b) $y'(3+4y) = e^{-x} - e^{x}$, $y(0) = 1$

$$\frac{1}{8}(4y+3)^2 = -e^{-x} - e^{x} + c$$

$$(4y+3)^2 = -8e^{-x} - 8e^{x} + c'$$

$$y = \frac{\sqrt{-8e^{-x} - 8e^{x} + c'} - 3}{4}$$

$$y(0) = \frac{1}{4}\left(\underbrace{\sqrt{-8-8+c} - 3}_{4}\right)$$

$$\sqrt{c-16} - 3 = 4$$

$$\sqrt{c-16} = 7 \quad \Rightarrow \quad c - 16 = 49$$

$$c = 65$$

$$\underline{\underline{y = -\frac{3}{4} + \frac{\sqrt{65 - 8e^{-x} - 8e^{x}}}{4}}}$$

2.3 (67)

a) $y' + y = \cos(x) + \sin(x)$

① Solution générale de l'ED sans second membre :

$y' = -y$

$\frac{y'}{y} = -1 \qquad \ln|y| = -x + C$

$$\underline{y = Ke^{-x}}$$

② solution particulière : facteur intégrant : $e^{\int dx} = e^{x}$

$y'e^{x} + ye^{x} = e^{x}\cos(x) + e^{x}\sin(x)$

$ye^{x} = e^{x}\sin(x) + C$

$y = \sin(x) + Ce^{-x}$

b) $xy' - y = (x-1)e^{x}$

① $xy' - y = 0$

$y' = \frac{1}{x} y \Rightarrow \frac{y}{y'} = \frac{1}{x}$

$\Rightarrow \ln|y| = \ln|a| + C$

$y = Kx$

② $y' - \frac{1}{x}y = \frac{x-1}{x} e^{x}$

Facteur intégrant : $e^{-\int \frac{1}{x} dx} = e^{-\ln(x) + C} = \frac{1}{x}$

L'équation devient :

$$\underbrace{\frac{1}{x}y' - \frac{1}{x^2}y}_{\left(\frac{1}{x}y\right)'} = \frac{x-1}{x^2} e^{x}$$

Intégrons $\frac{x-1}{x^2}e^x$

$$\int \frac{e^x}{x}dx - \underbrace{\int \frac{e^x}{x^2}dx}_{\text{intégrons par partie}} = \int \frac{e^x}{x}dx - \left(-\frac{e^x}{x} + \int \frac{e^x}{x}dx\right)$$

$v = e^x \qquad v' = e^x dx$

$u = -\frac{1}{x} \qquad u' = +\frac{1}{x^2}dx$

$$= \int \frac{e^x}{x}dx + \frac{e^x}{x} - \int \frac{e^x}{x}dx = \frac{e^x}{x} + C$$

donc $\int \frac{x-1}{x^2}e^x dx = \frac{e^x}{x}$

Ainsi $\frac{1}{x}y = \frac{e^x}{x} \Rightarrow y = e^x$

Solution $y = Cx + e^x$

Take the integral:

$$\int \frac{e^x (x-1)}{x^2} \, dx$$

Expand $\frac{e^x (x-1)}{x^2}$ giving $\frac{e^x}{x} - \frac{e^x}{x^2}$:

$$= \int \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) dx$$

Integrate the sum term by term:

$$= \int -\frac{e^x}{x^2} \, dx + \int \frac{e^x}{x} \, dx$$

For the integrand $-\frac{e^x}{x^2}$, integrate by parts, $\int f \, dg = f g - \int g \, df$, where

$f = e^x, \quad dg = -\frac{1}{x^2} \, dx,$

$df = e^x \, dx, \quad g = \frac{1}{x}$:

$$= \frac{e^x}{x} - \int \frac{e^x}{x} \, dx + \int \frac{e^x}{x} \, dx$$

Simplify:

Answer:

$$= \frac{e^x}{x} + \text{constant}$$

Computed by Wolfram|Alpha

2.3 (67)

c) $y' \cos(x) + y \sin(x) = 1 \quad (*)$

$$y' + \tan(x)\, y = \frac{1}{\cos(x)} \quad (**)$$

(SSM) : $$\frac{y'}{y} = -\tan(x)$$

$$\ln|y| = \ln|\cos(x)| + C$$

$$y = C \cdot \cos(x)$$

(Avec SM) : facteur intégrant :

$$e^{\int \tan(x)\,dx} = e^{-\ln(\cos(x))} = \frac{1}{\cos(x)}$$

Ainsi (**) devient :

$$\frac{1}{\cos(x)} y' + \frac{\sin(x)}{\cos^2(x)} y = \frac{1}{\cos^2(x)}$$

$$\left(\frac{1}{\cos(x)} y\right)' = (\tan(x))' \Rightarrow \frac{y}{\cos(x)} = \tan(x)$$

$$\Rightarrow y = \sin(x)$$

Finalement : $y = \sin(x) + C\cos(x)$

d) $xy' - 2y = x^3$

$$y' - \frac{2}{x} y = x^2$$

① EDSSM : $y' = \frac{2}{x} y \Rightarrow \frac{y'}{y} = \frac{2}{x} \Rightarrow \ln|y| = \ln(x^2) + C$

$$y = C x^2$$

② EDASS : fi $e^{\int -\frac{2}{x} dx} = e^{-2\ln(x)} = \frac{1}{x^2}$

$$y' \cdot \frac{1}{x^2} - \frac{2}{x^3} \cdot y = 1$$

$$\left(\frac{1}{x^2} y\right)' = (x)' \Rightarrow y = x^3$$

Finalement : $\underline{\underline{y = Cx^2 + x^3}}$

2.3 (67)

e) $y' + 2xy = 4x$

(1) EDSSM : $y' + 2xy = 0$

$y' = -2xy \Leftrightarrow \frac{y'}{y} = -2x$

$\ln(y) = -x^2 + C \Rightarrow y = Ke^{-x^2}$

(2) EDASM : fi $e^{\int 2x} = e^{x^2}$

$y' e^{x^2} + 2x e^{x^2} y = e^{x^2} \cdot 4x$

$(e^{x^2} y)' = 2(e^{x^2})'$

$\Rightarrow y = 2$

Solution : $\underline{y = Ke^{-x^2} + 2}$

f) $y' + y = 5\sin(2x)$

① EDSSM : $y' = -y \Leftrightarrow \frac{y'}{y} = -1$

$\Leftrightarrow \ln(y) = -x + C$

$y = K e^{-x}$

② EDASM ; fi $e^{\int dx} = e^{x}$

$$\underbrace{y'e^{x} + ye^{x}}_{(ye^{x})'} = 5e^{x}\sin(2x)$$

$$(ye^{x})' = 5\int e^{x}\sin(2x)\,dx$$

Par partie :

$$\int e^{x}\sin(2x)\,dx = e^{x}\sin(2x) - 2\int e^{x}\cos(2x)\,dx$$

$u = e^{x}$ $\quad u' = e^{x}dx$ $\quad u_1 = e^{x}$ $\quad u_1' = e^{x}dx$

$v = \sin(2x)$ $\quad v' = 2\cos(2x)\,dx$ $\quad v_1 = \cos(2x)$ $\quad v_1' = -2\sin(2x)\,dx$

$$= e^{x}\sin(2x) - 2\left[e^{x}\cos(2x) + 2\int e^{x}\sin(2x)\,dx\right]$$

Donc $\int e^{x}\sin(2x)\,dx = e^{x}\sin(2x) - 2e^{x}\cos(2x) - 4\int e^{x}\sin(2x)\,dx$

$$5\int e^{x}\sin(2x)\,dx = e^{x}\sin(2x) - 2e^{x}\cos(2x)$$

Donc $ye^{x} = e^{x}\sin(2x) - 2e^{x}\cos(2x)$

$$y = \sin(2x) - 2\cos(2x)$$

Finalement : $\underline{\underline{y = Ke^{-x} + \sin(2x) - 2\cos(2x)}}$

2.4

a) $y' - y = 2x e^{2x}$

① $\frac{y'}{y} = 1$ $\quad \ln(y) = x + c \Rightarrow y = K e^{x}$

② fi $e^{-\int dx} = e^{-x}$

$$y' e^{-x} - y e^{-x} = 2x e^{x}$$

$$(y e^{-x})' = 2\int x e^{x} dx$$

$$\int x e^{x} dx = x e^{x} - \int e^{x} dx = x e^{x} - e^{x}$$

$u = e^{x}$ $\quad u' = e^{x} dx$
$v = x$ $\quad v' = dx$

Donc $\quad y e^{-x} = 2x e^{x} - 2e^{x}$

$$y = 2x e^{2x} - 2e^{2x}$$

$y = K e^{x} + 2x e^{2x} - 2e^{2x}$ $\qquad y(0) = 1$

$1 = K + 0 - 2 \Rightarrow K = 3$

Finalement $\underline{y = 3e^{x} + 2e^{2x}(x-1)}$

b) $y' + \frac{2}{x} y = \frac{1}{x^2} \cos(x)$ $\qquad y(\pi) = 0, \; x > 0$

① $y' = -\frac{2}{x} y \Rightarrow \frac{y'}{y} = \frac{-2}{x}$

$\ln(y) = -2 \ln(x) + C \Rightarrow y = K \frac{1}{x^2}$

② Fi : $e^{\int \frac{2}{x} dx} = e^{2 \ln(x)} = x^2$

$x^2 y' + 2xy = \cos(x)$

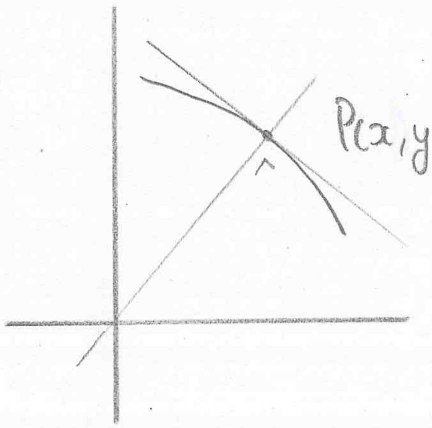
$(x^2 y)' = (\sin(x))' \Rightarrow y = \frac{\sin(x)}{x^2}$

Solution $y = K \cdot \frac{1}{x^2} + \frac{\sin(x)}{x^2}$

$y(\pi) = 0 \Rightarrow \quad 0 = \frac{K}{\pi^2} + 0 \Rightarrow K = 0$

Finalement : $y = \frac{\sin(x)}{x^2}$

2.7 (G7)



$$\frac{-1}{y'} = \frac{y}{x} \Rightarrow -y'y = x$$

Résolvons $y'y = -x$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$y^2 = C_2 - x^2 \Rightarrow y = \sqrt{C_2 - x^2}$$

ou $\underline{\underline{x^2 + y^2 = C_2}}$

2.8 (67)

$$y' = kx$$

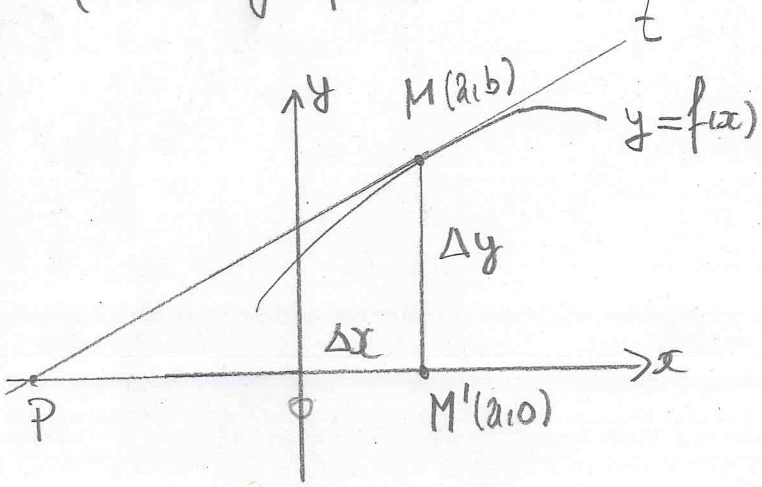
$$y = \frac{1}{2}x^2 + C$$

2,9 (67)

$$\boxed{\tan(\alpha) = \frac{\Delta y}{\Delta x}}$$

(Π) : $y = f(x)$

$\tan(\alpha) = y'$



$$\overrightarrow{M'P} = K\,\overrightarrow{M'O}$$

(t) : $f(x) = f'(a)(x-a) + f(a)$

$$\frac{y}{y'} = Kx \Rightarrow \frac{y'}{y} = \frac{1}{Kx}$$

$$\ln|y| = \frac{1}{K}\ln|x| + C$$

$$y = x^{1/K + C}$$

$$\Rightarrow \boxed{y^K = C_0 x}$$

2.11 (page 67)

a) Soit $y(t)$ la quantité qui reste.

$y' = -ky$ est l'éq. diff pour $k > 0$

Résolvons-la : $\frac{y'}{y} = -k \Rightarrow y = y_0 e^{-kt}$ avec $k > 0$

et $y(0) = y_0$

b) $y' = -ky + I$

$y' + ky = I$ f.i. e^{kt}

$y' e^{kt} + k e^{kt} y = I e^{kt}$

$y e^{kt} = \frac{I}{k} e^{kt} + C$

$y = Ce^{-kt} + \frac{I}{k}$

avec $y(0) = 0 \Rightarrow C + \frac{I}{k} = 0 \Rightarrow C = \frac{-I}{k}$

$$y = \frac{I}{k}\left(1 - e^{-kt}\right)$$

$$\lim_{t \to +\infty} y = \frac{I}{k}$$

c) demi-vie : après 2 heures, il reste la moitié de la substance :

A partir de $y = y_0 e^{-kt}$, on a

$e^{-120k} = \frac{1}{2} \Rightarrow k = \frac{\ln(2)}{120}$

Comme $\frac{I}{k} = 100$, alors $I = 100k = \frac{100}{120}\ln(2) = \frac{5\ln(2)}{6}$

2.12 (68)

L'équation différentielle : $y' = x + y$

Résolvons cette équation diff :

$$y' - y = x$$

1) Solution générale de l'ED sans second membre :

$$y' = y$$

$$\frac{y'}{y} = 1 \Rightarrow \ln|y| = x + c$$

$$y = Ke^{x}$$

2) Solution particulière : facteur intégrant : $e^{\int -dx} = e^{-x}$

$$\underbrace{y'e^{-x} - ye^{-x}}_{(ye^{-x})'} = xe^{-x}$$

$$(ye^{-x})' = xe^{-x}$$

Intégrons xe^{-x} par parties :

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$

$$= e^{-x}(-x-1)$$

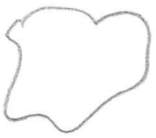
$u = -e^{-x}$ $\quad u' = e^{-x}dx$

$v = x$ $\quad v' = dx$

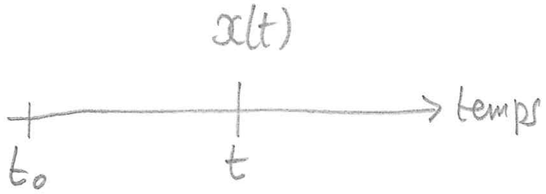
Ainsi : $ye^{-x} = e^{-x}(-x-1) \Rightarrow y = -x - 1$

Finalement : $y = Ke^{x} - x - 1$

2.13



n molécules



$x(t)$ nombre de molécules transformées au temps t , $t \geq t_0$

- $x(t_0) = 0$
- $x'(t) = k(n - x(t))$

$x'(t) + kx(t) = kn$

1) $x'(t) = -kx(t)$

$$\frac{x'(t)}{x(t)} = -k \Rightarrow \ln(x(t)) = -kt + a$$

$$x(t) = be^{-kt}$$

2) f.i: $e^{\int k\,dt} = e^{kt}$

$$x'(t)e^{kt} + ke^{kt}x(t) = kne^{kt}$$

$$(x(t)e^{kt})' = (ne^{kt})'$$

$$x(t)e^{kt} = ne^{kt}$$

$$x(t) = n$$

$$x(t) = be^{-kt} + n$$

Conditions initiales: $x(t_0) = 0$

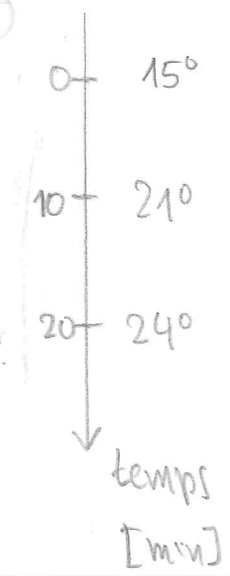
$$be^{-kt_0} + n = 0$$

$$b = -ne^{kt_0}$$

Ainsi

Finalement : $x(t) = n - ne^{kt_0}e^{-kt} = n\left(1 - e^{k(t_0 - t)}\right)$

2.14 (page 68) Loi de refroidissement de Newton



tem. initiale objet : T

$$\dot{T}(t) = K(T(t) - m)$$

$$T' - KT = -Km$$

① $T' = KT \Rightarrow \frac{T'}{T} = K \Rightarrow \ln(T) = Kt + a$

$$T = b e^{Kt}$$

② fi $e^{\int -K dt} = -e^{-Kt}$

$$(-e^{-Kt})T' - K(-e^{-Kt})T = -Km(-e^{-Kt})$$

$$(-e^{-Kt}T)' = (me^{-Kt})'$$

$$T = m$$

$$T(t) = b e^{Kt} + m$$

Il faut ainsi déterminer m.

en 0: $b + m = 15$

en 10: $b e^{10K} + m = 21$

en 20: $b e^{20K} + m = 24$

Résolvons ce système.

Posons $\alpha = e^{10k}$. Le système devient :

$$\begin{cases} b\alpha + 15 - b = 21 \\ b\alpha^2 + 15 - b = 24 \\ b + m = 15 \end{cases}$$

$$\begin{cases} b + m = 15 \\ b(\alpha - 1) = 6 \\ b(\alpha^2 - 1) = 9 \end{cases} \Rightarrow \alpha - 1 = \frac{6}{b}$$

$$\begin{cases} b + m = 15 \\ b(\alpha - 1) = 6 \\ \cancel{b}(\alpha + 1)\frac{6}{\cancel{b}} = 9 \end{cases} \Rightarrow \alpha + 1 = \frac{9}{6} \Rightarrow \alpha = \frac{3}{2} - 1 = \frac{1}{2}$$

Calculons k : $e^{10k} = \frac{1}{2} \Rightarrow k = \frac{-\ln(2)}{10}$.

$$\begin{cases} b + m = 15 \\ b\, e^{-\ln(2)} + m = 21 \\ b\, e^{-\ln(2)\cdot 2} + m = 24 \end{cases} \Rightarrow \begin{cases} b + m = 15 \\ b \cdot \frac{1}{2} + m = 21 \quad \cdot 1 \\ b \cdot \frac{1}{4} + m = 24 \quad \cdot(-1) \end{cases}$$

$$b + m = 15$$

$$\frac{1}{4} b = -3 \Rightarrow \boxed{b = -12}$$

et finalement $\underline{m = 27}$

2.16 (page 68)

$V' = KS$

On sait que $V = \frac{4}{3}\pi R^3 \Rightarrow R = \sqrt[3]{\frac{3V}{4\pi}} = \alpha V^{1/3}$

Ainsi $V' = K \cdot 4\pi (\alpha V^{1/3})^2$

$V' = \underbrace{K \cdot 4\pi \cdot \alpha^2}_{} \cdot V^{2/3}$

$\Rightarrow V' = \beta V^{2/3}$

$\Rightarrow \underbrace{V' \cdot V^{-\frac{2}{3}}}_{} = \beta$

$3\left(V^{\frac{1}{3}}\right)' = (\beta t + \gamma)'$

$\Rightarrow 3V^{1/3} = \beta t + \gamma$

$V^{1/3} = \beta_0 t + \gamma_0$

$V = (\beta_0 t + \gamma_0)^3$

Si $t = 0$: $V_0 = (\gamma_0)^3 \Rightarrow \gamma_0 = V_0^{1/3}$

$$\boxed{V = \left(\beta_0 t + V_0^{1/3}\right)^3}$$