34. Calculer les intégrales suivantes à 10^{-4} près.

b) $\displaystyle\int_0^1 e^{-x^2}\,dx$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{k}}{k!} + \dots$$

$$\Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + (-1)^{k}\,\frac{x^{2k}}{k!} + \dots$$

On calcule l'intégrale :

$$\int_0^1 e^{-x^2}\,dx = \int\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + (-1)^{2k}\,\frac{x^{2k}}{k!} + \dots\right)dx$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^{2k}}{k!}\cdot\int_0^1 x^{2k}\,dx$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^{2k}}{k!}\cdot\left[\frac{x^{2k+1}}{2k+1}\right]_0^1$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^{2k}}{k!}\cdot\frac{1}{2k+1} = \sum_{k=0}^{+\infty} (-1)^{2k}\,\frac{1}{k!\,(2k+1)}$$

$$\int_0^1 e^{-x^2}\,dx = \underset{k=0}{1} - \underset{1}{\frac{1}{3}} + \underset{2}{\frac{1}{10}} - \underset{3}{\frac{1}{42}} + \underset{4}{\frac{1}{216}} - \underset{5}{\frac{1}{1320}} + \underset{6}{\frac{1}{9360}} \cong 0,7468$$

$$\int_0^1 e^{-x^2}\,dx \cong 0,7468 - \frac{1}{75600} \cong 0,7468 - 0,00013228$$

A quatre décimales : $\int_0^1 e^{-x^2}\, dx \cong 0{,}7468$

e) $\int_0^{\frac{1}{4}} \sqrt{1+x^3}\, dx$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \frac{15}{384}x^4 + \dots$$

$$\Rightarrow \sqrt{1+x^3} = 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{3}{48}x^9 - \frac{15}{384}x^{12} + \dots$$

$$\Rightarrow \int_0^{\frac{1}{4}} \sqrt{1+x^3}\, dx \cong x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \frac{3}{480}x^{10} + \dots$$

$$\cong \left[x + \frac{1}{8}x^4 - \frac{1}{56}x^7\right]_0^{1/4} \cong \frac{1}{4} + \frac{1}{8}\cdot\left(\frac{1}{4}\right)^4 - \frac{1}{56}\cdot\left(\frac{1}{4}\right)^7$$

$$= \underbrace{\frac{1}{4} + \frac{1}{2048}} - \frac{1}{917'504}$$

$$= 0{,}2505 - \frac{1}{917'504} = 0{,}2505$$

$$\Rightarrow \int_0^{1/4} \sqrt{1+x^3}\, dx \cong 0{,}2505$$

g) $\int_0^{\frac{1}{2}} \arctan(x^2)\, dx$

$$\arctan(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

$$\Rightarrow \arctan(x^2) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} (x^2)^{2k+1} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} x^{4k+2}$$

$$\Rightarrow \int_0^{1/2} \arctan(x^2)\,dx = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} \cdot \frac{1}{4k+3} x^{4k+3} \Big|_0^{1/2}$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)(4k+3)} \cdot \frac{1}{2^{4k+3}}$$

$$= \frac{1}{3\cdot 2^3} - \frac{1}{3\cdot 7\cdot 2^7} + \frac{1}{5\cdot 11\cdot 2^{11}} - \frac{1}{7\cdot 15\cdot 2^{15}} + \dots$$

Arrondi à 4 décimales :

$$\frac{1}{3\cdot 2^3} - \frac{1}{3\cdot 7\cdot 2^7} \cong 0,0413$$

$$\frac{1}{3\cdot 2^3} - \frac{1}{3\cdot 7\cdot 2^7} + \frac{1}{5\cdot 11\cdot 2^{11}} \cong 0,0413$$

$$\Rightarrow \int_0^{1/2} \arctan(x^2)\,dx \cong 0,0413$$

d) $$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

$$\frac{\sin(x)}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3!} + \frac{x^{9/2}}{5!} - \dots + (-1)^k \frac{x^{2k+\frac{1}{2}}}{(2k+1)!}$$

$$\int_0^1 \frac{\sin(x)}{\sqrt{x}}\,dx = \int_0^1 \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+\frac{1}{2}}}{(2k+1)!} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+\frac{1}{2}}\,dx$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{2k+\frac{3}{2}} \left[x^{2k+\frac{3}{2}}\right]_0^1 = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} \frac{2}{4k+3}$$

$$\int_0^1 \frac{\sin(x)}{\sqrt{x}}\,dx = \underbrace{\underbrace{\overset{k=0}{\frac{2}{3}} - \overset{1}{\frac{1}{3!}} \cdot \frac{2}{7} + \overset{2}{\frac{1}{5!}} \cdot \frac{2}{11}}_{\cong\ 0{,}6205} - \frac{1}{9!} \cdot \frac{2}{15}}_{\cong\ 0{,}6205}$$