Révision équations différentielles

Exercice 1

Résoudre les équations différentielles suivantes :

a) $x\,dx + y\,dy = 0$

b) $x\,dx - y^3\,dy = 0$

c) $x\,dx + \frac{1}{y}\,dy = 0$

d) $y' = \frac{y}{x^2}$

e) $y' = \frac{x^2}{y}$

f) $y' + y^2 \sin(x) = 0$

Exercice 2

Résoudre l'équation différentielle suivante :

$$y' = 2\,y^2 + x\,y^2$$

avec la condition initiale $y(0) = 1$.

Exercice 3

Résoudre les équations différentielles suivantes :

a) $y' - 2\,x\,y = x$

b) $y' + y = \sin(x)$

c) $y' + x^2\,y = x^2$

d) $N' + \frac{1}{t}\,N = t$ avec $N(2) = 8$

Exercice 4

Résoudre les équations différentielles suivantes :

a) $y'' - 2y' + y = 0$

b) $4\,y'' - 4\,y' - 3\,y = 0$

c) $y'' - 2\,y' + 10\,y = 0$

d) $y'' - 2\,y' - 3\,y = 3\,e^{2t}$

e) $y' + 2\,y' + 5\,y = 3\,\sin(2t)$

f) $y'' - 2y' - 3y = -3\,t\,e^{-t}$

Exercice 5

Résoudre l'équation différentielle suivante :

$$y'' - 2\,y' - 3\,y = 3\,t\,e^t$$

avec la condition initiale $y(0) = 1$ et $y'(0) = 0$.

Exercice 1

a) $y\,dy = -x\,dx$

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c \quad \Rightarrow \quad \underline{x^2 + y^2 = c}$ (solution implicite)

b) $y^3\,dy = x\,dx$

$\frac{1}{4}y^4 = \frac{1}{2}x^2 + c \quad \Rightarrow \quad \underline{y^4 - 2x^2 = c}$ (solution implicite)

c) $\frac{1}{y}\,dy = -x\,dx$

$\ln|y| = -\frac{1}{2}x^2 + c \quad \Rightarrow \quad \underline{y = Ke^{-\frac{x^2}{2}}}$

d) $\frac{y'}{y} = \frac{1}{x^2}$

$\ln|y| = \frac{-1}{x} + c \quad \Rightarrow \quad \underline{y = Ke^{-\frac{1}{x}}}$

e) $y'y = x^2$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 + c$

$y^2 = \frac{2}{3}x^3 + c \quad \Rightarrow \quad \underline{y = \pm\sqrt{\frac{2}{3}x^3 + c}}$

f) $\frac{y'}{y^2} = -\sin(x)$

$-\frac{1}{y} = \cos(x) + c \quad \Rightarrow \quad \underline{y = \frac{-1}{\cos(x) + c}}$

Exercice 2

$$y' = 2\,y^2 + x\,y^2$$

$$y' = y^2\,(x+2) \Rightarrow \frac{y'}{y^2} = x+2 \Rightarrow \frac{-1}{y} = \frac{1}{2}x^2 + 2x + C$$

$$\Rightarrow y = \frac{-1}{\frac{1}{2}x^2 + 2x + C}$$

Avec $y(0) = 1$: $\quad 1 = \dfrac{-1}{C} \Rightarrow C = -1$

$$y = \frac{-1}{\frac{1}{2}x^2 + 2x - 1} \quad \text{ou} \quad \boxed{y = \frac{-2}{x^2 + 4x - 2}}$$

Exercice 3

a) $y' - 2xy = x$

1) Sol. éq. homogène : $y_0 = ce^{-\int -2x\,dx} = \underline{ce^{x^2}}$

2) Sol. particulière : $f_i = e^{\int -2x\,dx} = e^{-x^2}$

$$\underbrace{e^{-x^2}y' - 2xe^{-x^2}y} = x \cdot e^{-x^2}$$

$$\left(e^{-x^2}y\right)' = \left(-\frac{1}{2}e^{-x^2}\right)'$$

$$e^{-x^2}y = -\frac{1}{2}e^{-x^2} \Rightarrow \underline{y_p = -\frac{1}{2}}$$

Solution : $\underline{y = y_0 + y_p = ce^{x^2} - \frac{1}{2}}$

b) $y' + y = \sin(x)$

1) $\underline{y_0} = ce^{-\int 1\,dx} = \underline{ce^{-x}}$

2) $y_p = a\sin(x) + b\cos(x)$

$y_p' = a\cos(x) - b\sin(x)$

$\Rightarrow y_p' + y = (a-b)\sin(x) + (a+b)\cos(x)$

$$\Rightarrow \begin{cases} a - b = 1 \\ a + b = 0 \end{cases} \Leftrightarrow \begin{cases} 2a = 1 \\ 2b = -1 \end{cases} \Leftrightarrow \begin{matrix} a = \frac{1}{2} \\ b = -\frac{1}{2} \end{matrix}$$

$\underline{y_p = \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)}$

Sol : $\underline{y = ce^{-x} + \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)}$

c) $y' + x^2 y = x^2$

1) $\underline{y_0} = c\, e^{-\int x^2 dx} = \underline{c\, e^{-\frac{x^3}{3}}}$

2) $fi = e^{\int x^2 dx} = e^{\frac{1}{3}x^3}$

$$\underbrace{e^{\frac{1}{3}x^3} \cdot y' + x^2 e^{\frac{1}{3}x^3} y}_{\left(e^{\frac{1}{3}x^3} y\right)'} = x^2 e^{\frac{1}{3}x^3}$$

$$\left(e^{\frac{1}{3}x^3} y\right)' = \left(e^{\frac{1}{3}x^3}\right)' \Rightarrow \underline{y_p = 1}$$

$$\underline{y = C\, e^{-\frac{x^3}{3}} + 1}$$

d) $N' + \frac{1}{t} N = t$ avec $N(2) = 8$

1) $\underline{N_0} = c\, e^{-\int \frac{1}{t} dt} = c\, e^{-\ln(t)} = \underline{\frac{c}{t}}$

2) $fi = e^{\int \frac{1}{t} dt} = e^{\ln(t)} = t$

$$\underbrace{t N' + N}_{(tN)'} = t^2$$

$$(tN)' = \left(\frac{1}{3}t^3\right)' \Rightarrow \underline{N_p = \frac{1}{3}t^2}$$

$$\underline{N(t) = \frac{c}{t} + \frac{1}{3}t^2}$$

3) $N(2) = 8 \Rightarrow 8 = \frac{c}{2} + \frac{1}{3} \cdot 4 \Rightarrow c = 2\left(8 - \frac{4}{3}\right) = \frac{40}{3}$

$$\Rightarrow \underline{N(t)} = \frac{40}{3} \cdot \frac{1}{t} + \frac{1}{3}t^2 = \underline{\frac{1}{3}\left(t^2 + \frac{40}{t}\right)}$$

Exercice 4

a) $y'' - 2y' + y = 0$

éq. car : $r^2 - 2r + 1 = 0$; $r_1 = r_2 = 1$

Solution : $y = (c_1 x + c_2) e^{x}$

b) $4y'' - 4y' - 3y = 0$

éq. car : $4r^2 - 4r - 3 = 0$

$(2r-3)(2r+1) = 0$; $r_1 = \frac{3}{2}$, $r_2 = -\frac{1}{2}$

Solution : $y = c_1 e^{3/2 x} + c_2 e^{-1/2 x}$

c) $y'' - 2y' + 10y = 0$

éq. car : $r^2 - 2r + 10 = 0$

$\Delta = 4 - 40 = -36 = 36i^2 = (6i)^2$

$r_1 = 1 + 3i$ et $r_2 = 1 - 3i$

Solution : $y = e^{x} \left(c_1 \cos(3x) + c_2 \sin(3x) \right)$

d) $y'' - 2y' - 3y = 3e^{2t}$ $\left[r^2 - 2r - 3 = (r-3)(r+1) \right]$

$y_0 = c_1 e^{-t} + c_2 e^{3t}$

Sol. particulière : $p(x) = \alpha e^{2t}$ (CRM)

$p'(x) = 2\alpha e^{2t}$, $p''(x) = 4\alpha e^{2t}$

$\Rightarrow 4\alpha e^{2t} - 4\alpha e^{2t} - 3\alpha e^{2t} = 3e^{2t} \Rightarrow \alpha = -1$

Solution : $y = C_1 e^{-t} + C_2 e^{3t} - e^{2t}$

e) $y'' + 2y' + 5y = 3\sin(2t)$

éq. caractéristique : $r^2 + 2r + 5 = 0$

$\Delta = 4 - 20 = -16 = 16i^2 = (4i)^2$

$r_1 = -1 - 2i \quad ; \quad r_2 = -1 + 2i$

$y_0 = e^{-t}\left(C_1 \cos(2t) + C_2 \sin(2t)\right)$

Solution particulière : $p(t) = \alpha \sin(2t) + \beta \cos(2t)$ CRM

$p'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$

$p''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$

$p''(t) + 2p'(t) + 5p(t) = (-4\alpha - 4\beta + 5\alpha)\sin(2t) + (-4\beta + 4\alpha + 5\beta)\cos(2t)$

$$\begin{cases} \alpha - 4\beta = 3 \\ 4\alpha + \beta = 0 \end{cases} \left|\begin{array}{c} \beta \\ \cdot 1 \\ 4 \end{array}\right|\begin{array}{c} \alpha \\ \cdot(-4) \\ \cdot 1 \end{array} \Rightarrow \begin{cases} 17\alpha = 3 \\ 17\beta = -12 \end{cases} \Rightarrow \begin{cases} \alpha = 3/17 \\ \beta = -12/17 \end{cases}$$

$y = e^{-t}\left(C_1 \cos(2t) + C_2 \sin(2t)\right) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$

f) $y'' - 2y' - 3y = -3t e^{-t}$

1) $y_0 = C_1 e^{-t} + C_2 e^{3t}$

2.1) Solution particulière : $p(t) = (at+b)e^{-t}$ (cours)

$p'(t) = a e^{-t} - (at+b)e^{-t} = (-at + a - b)e^{-t}$

$$p''(t) = -a\,e^{-t} - (-at+a-b)e^{-t} = (at-2a+b)\,e^{-t}$$

En substituant :

$$[at - 2a + b + 2at - 2a + 2b - 3at - 3b]\,e^{-t} = -3t\,e^{-t}$$

$p(t) = (at+b)\,e^{-t}$ ne convient pas !

2.2) Solution particulière bis : $p(t) = (at^2+bt)\,e^{-t}$ $p''-2p'-3p$

$$p' = (2at+b)e^{-t} - (at^2+bt)e^{-t}$$

$$p'' = 2a\,e^{-t} - 2(2at+b)e^{-t} + (at^2+bt)e^{-t}$$

$$2a e^{-t} - 2(2at+b)e^{-t} + (at^2+bt)e^{-t} - 2(2at+b)e^{-t} + 2(at^2+bt)e^{-t} - 3(at^2+bt)e^{-t}$$
$$= -3t\,e^{-t}$$

$$\Rightarrow 2a - 4at - 2b + at^2 + bt - 2at - 2b + 2at^2 + 2bt - 3at^2 - 3bt = -3t\,e^{-t}$$

$$2a - 4b - 6at + bt - 2at + 2bt - 3bt = -3t$$

$$\begin{cases} 2a - 4b = 0 \\ -8a = -3 \end{cases} \Rightarrow \quad \begin{aligned} b &= \tfrac{1}{2}a = \tfrac{3}{16} \\ a &= \tfrac{3}{8} \end{aligned}$$

$$p(t) = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}$$

$$\boxed{y = C_1 e^{-t} + C_2 e^{3t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}}$$

Exercice 5

$$y'' - 2y' - 3y = 3te^t$$

1) $y_0 = C_1 e^{-t} + C_2 e^{3t}$

2) Solution particulière : $p(t) = (at+b)e^t$

$$p'(t) = ae^t + (at+b)e^t = (at+a+b)e^t$$
$$p''(t) = ae^t + (at+a+b)e^t = (at+2a+b)e^t$$

$$at + \cancel{2a} + b - 2at - \cancel{2a} - 2b - 3at - 3b = 3t$$

$$-4at - 4b = 3t \Rightarrow \begin{cases} -4a = 3 \\ b = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{3}{4} \\ b = 0 \end{cases} \Rightarrow p(t) = \frac{-3}{4} t e^t$$

3) $y = C_1 e^{-t} + C_2 e^{3t} - \frac{3}{4} t e^t$

4) $y(0) = 1$ et $y'(0) = 0$.

$$y = C_1 e^{-t} + C_2 e^{3t} - \frac{3}{4} t e^t \;;\; y(0) = C_1 + C_2 = 1$$

$$y' = -C_1 e^{-t} + 3C_2 e^{3t} - \frac{3}{4} e^t - \frac{3}{4} t e^t \;;\; y'(0) = -C_1 + 3C_2 - \frac{3}{4} = 0$$

d'où $\begin{cases} C_1 + C_2 = 1 \\ -C_1 + 3C_2 = \frac{3}{4} \end{cases} \Rightarrow \begin{cases} 4C_2 = \frac{7}{4} \\ 4C_1 = \frac{9}{4} \end{cases} \Rightarrow \begin{cases} C_2 = \frac{7}{16} \\ C_1 = \frac{9}{16} \end{cases}$

Finalement : $\boxed{y = \frac{9}{16} e^{-t} + \frac{7}{16} e^{3t} - \frac{3}{4} t e^t}$